

On Period Relations for Automorphic L -functions I

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Abstract

Since Euler we know that the Riemann ζ -function satisfies the relation

$$\zeta(2k) \in (2\pi i)^{2k} \mathbf{Q}, \quad k \geq 0.$$

Building on the author's recent results on rational structures on unitary representations of real reductive groups, we generalize Euler's result to special values of certain automorphic L -functions. In particular we prove new period relations for those L -functions and show that they are in accordance with Deligne's Conjecture.

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Introduction

In his solution to the Basel Problem, Euler showed in [12] that the Riemann ζ -function has the remarkable property that

$$\zeta(2k) \in (2\pi i)^{2k} \mathbf{Q}^\times,$$

for all integers $k \geq 1$. Thanks to the work of Riemann [39] we also know that this is valid even for the (non-critical) value $k = 0$.

Euler's result found many generalizations, to Dedekind ζ -functions, Dirichlet- and Hecke L -functions and L -functions attached to modular cusp forms [35, 42], and more generally Hilbert modular forms [36, 43, 44]. However, outside of $\mathrm{GL}(2)$ (and products thereof), no such relation is known in the higher rank case, although we expect this to be true (cf. [11]).

Our main result is the following. For each irreducible regular algebraic cuspidal automorphic representations Π of $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$, whose weight is critical in the sense of section 4.1, and each $k \in \mathbf{Z}$ such that $s = \frac{1}{2} + k$ is critical for $L(s, \Pi)$ in the sense of Deligne, we have

$$L\left(\frac{1}{2} + k, \Pi\right) \in (2\pi i)^{m \cdot k} \cdot \mathbf{Q}(\Pi) \cdot \Omega_{(-1)^k}(\Pi) \quad (1)$$

with two complex constants $\Omega_\pm(\Pi) \in \mathbf{C}^\times$ independent of k , $\mathbf{Q}(\Pi)$ a number field.

To be more concrete, let F/\mathbf{Q} be a totally real number field of degree r_F and define

$$G_n := \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}(n).$$

Let Π be an irreducible regular algebraic cuspidal automorphic representation of

$$G = G_{n+1} \times G_n.$$

We may decompose $\Pi = \pi \hat{\otimes} \sigma$ with irreducible cuspidal representations π and σ of G_{n+1} and G_n , respectively, and write $L(s, \Pi) = L(s, \pi \times \sigma)$ for the the

incomplete Rankin-Selberg L -function attached to Π , i.e. $L(s, \pi \times \sigma)$ is an Euler product over all finite places without the Γ factors.

To such a Π we may conjecturally attach a motive $M(\Pi)$ with the property that

$$L(s - \frac{2n-1}{2}, \Pi) = L(s, M(\Pi)). \quad (2)$$

Assuming the existence of $M(\Pi)$, Deligne's Conjecture [11] predicts, among many other things, that (1) is true with $m = \frac{(n+1)n}{2}r_F$ (cf. section 5).

We have a well known cohomological definition for periods $\Omega_{\pm}(\Pi, s_0) \in \mathbf{C}^{\times}$ under mild conditions on the 'weight' of Π , depending on the critical s_0 under consideration, such that (1) holds where a priori the periods *vary* with k .

We know that the periods $\Omega_{\pm}(\Pi, s_0) \in \mathbf{C}^{\times}$ behave as expected under twists with finite order Hecke characters by [31, 22, 23] (even over arbitrary number fields [20, 24, 37]).

The contents of our Main Theorem (Theorem A below and Theorem 5.2 in the text) is that the predicted period relations for varying s_0 are indeed satisfied for $1 \leq n \leq 2$ for totally real F , and conditionally for $n \geq 3$ (cf. Conjecture 4.7).

Theorem A. *Assume $1 \leq n \leq 2$ or that Conjecture 4.7 is true for $n \geq 3$. Let F be a totally real number field of degree r_F over \mathbf{Q} , and (π, σ) be a pair regular algebraic irreducible cuspidal automorphic representation of $\mathrm{GL}_{n+1}(\mathbf{A}_F)$ and $\mathrm{GL}_n(\mathbf{A}_F)$ respectively. Assume that $\pi_{\infty} \widehat{\otimes} \sigma_{\infty}$ has non-trivial relative Lie algebra cohomology with coefficients in an irreducible rational $G_{n+1} \times G_n$ -module M , which we assume to be critical in the sense of section 4.1. Then there exist non-zero periods Ω_{\pm} , numbered by the $2^{[F:\mathbf{Q}]}$ characters \pm of $\pi_0(F_{\infty}^{\times})$, such that for each critical half integer $s_0 = \frac{1}{2} + j_0$ for the Rankin-Selberg L -function $L(s, \pi \times \sigma)$, and each finite order Hecke character*

$$\chi: F^{\times} \backslash \mathbf{A}_F^{\times} \rightarrow \mathbf{C}^{\times},$$

we have, in accordance with Deligne's Conjecture (cf. 5.1),

$$\frac{L(s_0, (\pi \times \sigma) \otimes \chi)}{G(\overline{\chi})^{\frac{(n+1)n}{2}} (2\pi i)^{j_0 r_F} \Omega_{(-1)^{j_0 \operatorname{sgn} \chi}}^{\frac{(n+1)n}{2}}} \in \mathbf{Q}(\pi, \sigma, \chi).$$

Furthermore the expression on the left hand side is $\operatorname{Aut}(\mathbf{C}/\mathbf{Q})$ -equivariant.

We remark that the unconditional case $n = 2$ corresponds to the degree $6r_F$ Rankin-Selberg L -function for $\mathrm{GL}_3 \times \mathrm{GL}_2$ over totally real number fields, and even over \mathbf{Q} our result is new in this case.

The case $n = 1$ corresponds to the case of Hilbert modular forms on $\mathrm{GL}(2)/F$. In this case we obtain a new entirely representation theoretic proof of the Manin-Shimura period relations alluded to above.

In principle our method works over arbitrary number fields as well. However we do not consider this case here, since all results would depend conditionally on the complex analogue of Conjecture 4.7 for all $n \geq 2$.

In [13] Grobner and Harris show in the imaginary quadratic case that there is indeed a relation between the automorphic periods $\Omega_{\pm}(\Pi, s_0)$ and honestly arithmetically defined periods. This result in turn has been exploited by Lin in her thesis to show that a similar period relation as in Theorem A holds in this context [33] for special values in the range of absolute convergence.

Recently Harder and Raghuram obtained partial results towards the period relations [15, 16, 17] by exploiting the structure of Eisenstein cohomology: Harder and Raghuram show that the ratio of the periods $\Omega_{\pm}(s_0, \Pi)$ and $\Omega_{\mp}(s_0 + 1, \Pi)$ is constant, which is a corollary of Theorem A (conditional for $n \geq 3$). We believe that a modification of our method here using the results on rational structures in section 2 allow for the computation of the desired ratio of periods in Harder and Raghuram's approach as well.

Our approach is a representation theoretic reinterpretation of the original argument for $\mathrm{GL}(2)$ and does not rely on the structure of Eisenstein cohomology. The main theme are the interplay of the rational structures introduced in [26] and the modular symbols that have been used in [31, 22, 23, 24, 37].

To be more specific, let us assume, that M in Theorems A is of highest weight μ , and write $A_{\mathbf{q}}(\mu)$ for the underlying (irreducible) (\mathfrak{g}, K) -module of Π_{∞} . It is a cohomologically induced standard module, as the notation suggests. As a (\mathfrak{g}, K^0) -module it decomposes into a direct sum

$$A_{\mathbf{q}}(\mu) = \bigoplus_{\varepsilon} \varepsilon A_{\mathbf{q}}^{\circ}(\mu) \quad (3)$$

of translates of an irreducible (\mathfrak{g}, K^0) -standard module $A_{\mathbf{q}}^{\circ}(\mu)$, ε running through a (dual) system of representatives in $G(\mathbf{Q})$ of the possible signs \pm in Theorems A. By Theorem 7.3 of [26] the field of definition of the module $A_{\mathbf{q}}(\mu)$ is the field of rationality $\mathbf{Q}(\mu)$ of M . The main result concerning the rationality of $A_{\mathbf{q}}^{\circ}(\mu)$ is (cf. Theorem 2.3 in the text),

Theorem B. *If $\sqrt{-1} \in \mathbf{Q}(\mu)$ then $A_{\mathbf{q}}^{\circ}(\mu)$ is defined over $\mathbf{Q}(\mu)$. Otherwise $A_{\mathbf{q}}^{\circ}(\mu)$ is defined over $\mathbf{Q}(\mu)$ if and only if*

$$4 \mid (n+1)n.$$

In all other cases it is defined over $\mathbf{Q}(\mu, \sqrt{-1})$.

In section 2 we prove a more general statement for tempered and unitarizable irreducible Casselman-Wallach representations of

$$\mathrm{GL}_{n_1}(\mathbf{R}) \times \mathrm{GL}_{n_2}(\mathbf{R}) \times \cdots \times \mathrm{GL}_{n_r}(\mathbf{R}),$$

with non-trivial Lie algebra cohomology, using the methods of [26]. The case $n_1 = 2n$ and $r = 1$ will be exploited in [27] in the context of Shalika models.

If $A_{\mathbf{q}}^{\circ}(\mu)$ is not defined over $\mathbf{Q}(\mu)$, complex conjugation, as an automorphism of $\mathbf{Q}(\mu)[\sqrt{-1}]/\mathbf{Q}(\mu)$, permutes the factors in the decomposition (3). In this sense (3) reflects the classical contribution of vectors from the holomorphic and

anti-holomorphic discrete series to Π_∞ in the case of a Hilbert modular form [36, 44].

The crucial observation is that the divisibility relation in Theorem B is equivalent to

$$i^{\frac{(n+1)n}{2}} \in \mathbf{Q} \quad (4)$$

which in turn matches the contribution of i to the period occurring in Theorem A up to the exponential factor r_F . We will see in the proof of Theorem 4.10, that this rationality pattern eventually explains the occurrence of i in the periods. The contribution of 2π to the periods corresponds to the ratio of the evaluation of the Γ -factors at different critical places.

We emphasize that the divisibility relation in Theorem B stems ultimately from the fact that a root system of type B_n or D_n admits the negated long Weyl element $-w_0$ as a non-trivial automorphism if and only if it is of type D_n with n odd. It is quite remarkable that this pattern matches Deligne's Conjecture.

In a nutshell, the general tools from [26] together with Theorem B overcome the fundamental problem that the minimal K^0 -types in the archimedean representation Π_∞ under consideration are no more one-dimensional and consequently cohomological vectors give rise to a higher dimensional subspace of the minimal K^0 -type rather than a single line. This becomes even more crucial when projecting those vectors to the different critical values.

We show that all those vectors lie in an appropriately normalized model of the representation over its field of rationality, and since the functionals are rational themselves, the period map preserves this rational structure and thus yields the desired period relations.

We emphasize that Theorem B alone is not enough to proof the desired period relation. The rationality properties of the local archimedean zeta integrals are crucial. In the Rankin-Selberg case we rely on the fundamental work of Kasten-Schmidt [30] and in the Shalika case we need Sun's Theorem 1.6 in [47], which in turn relies on the automatic continuity property of symmetric subgroups of Lie groups [2, 8], in combination with the Gelfand property of the pair $(\mathrm{GL}_{2n}(\mathbf{R}), \mathrm{GL}_n(\mathbf{R}) \times \mathrm{GL}_n(\mathbf{R}))$, cf. [1]. This will be exploited in [27].

Unfortunately Sun's ground-breaking results in [46] are not strong enough to deduce the necessary rationality properties of the archimedean zeta integral in the Rankin-Selberg case. The latter remains a difficult open problem in the Rankin-Selberg case for $n > 2$ and also in the presence of complex places.

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1 Notation and setup

We let $\overline{\mathbf{Q}} \subseteq \mathbf{C}$ denote the algebraic closure of \mathbf{Q} inside \mathbf{C} . If v is a place of a field E , we write E_v for its completion at v . If E/\mathbf{Q} is a number field, we write

$\mathbf{A}_E = \mathbf{A} \otimes_{\mathbf{Q}} E$ for the topological ring of adeles over E , where \mathbf{A} denotes the ring of adeles over \mathbf{Q} . We write $\mathbf{A}_E^{(\infty)} = \mathbf{A}^{(\infty)} \otimes_{\mathbf{Q}} E$ for the ring of finite adeles.

If E/F is a finite separable field extension, we write $\text{Res}_{E/F}$ for the functor of restriction of scalars à la Weil for the extension E/F , sending quasi-projective varieties over E to quasi-projective varieties over F . Since this functor preserves finite products, it sends group objects to group objects.

If G is an algebraic or a topological group, we denote its connected component of the identity by G^0 . We assume henceforth without further mention that the identity component of any linear algebraic group G is always *geometrically connected*, i.e. the geometrically connected component is already defined over the base field under consideration. Then its component group $\pi_0(G) = G/G^0$ is well defined independently of the base field under consideration. If \mathfrak{g} is the Lie algebra of G , we let $U(\mathfrak{g})$ denote its universal enveloping algebra. If G is an algebraic group defined over a field E , then \mathfrak{g} and $U(\mathfrak{g})$ are defined over E as well. We adopt the same notation in the context of Lie groups. In this case $U(\mathfrak{g})$ is defined over \mathbf{R} , but we usually implicitly consider its complexification in this case.

We write

$$X(G) = \text{Hom}(G, \text{GL}_1)$$

for the group of rational characters of G . If G is defined over a field E , we denote by $X_E(G)$ the subgroup of characters which are defined over E .

A superscript $(\cdot)^\vee$ on a (rational / admissible) representation of G , of $G(\mathbf{R})$ or a (\mathfrak{g}, K) -module denotes its (rational / admissible) dual.

We assume without loss of generality that all Haar measures on totally disconnected groups we consider have the property that the volumes of compact open subgroups are *rational numbers*.

In the body of the paper F/\mathbf{Q} denotes a totally real number field of degree r_F over \mathbf{Q} .

1.1 Algebraic groups

We are interested in products G of copies of

$$G_n = \text{Res}_{F/\mathbf{Q}}(\text{GL}_n).$$

We have a decomposition into a quasi-product

$$G_1 = \text{Res}_{F/\mathbf{Q}}(\text{GL}_1) = G_1^s \cdot G_1^{\text{an}}, \quad (5)$$

where G_1^s is the maximal \mathbf{Q} -split torus and G_1^{an} is the maximal \mathbf{Q} -anisotropic subtorus in G_1 . The latter is a quasi-complement of G_1^s in G_1 , i.e.

$$G_1^s \cap G_1^{\text{an}} \text{ is finite.}$$

The projection

$$p_s : G_1 \rightarrow G_1/G_1^{\text{an}} \cong \text{GL}_1$$

corresponds to the Norm character

$$N_{F/\mathbf{Q}} : F^\times \rightarrow \mathbf{Q}^\times,$$

and its composition with the determinant induces a character

$$\mathcal{N} := p_s \circ \text{Res}_{F/\mathbf{Q}} \det : G_n \rightarrow \text{GL}_1.$$

To describe its behavior on the real points, we introduce for each archimedean place v of F the local *sign character*

$$\text{sgn}_v : \text{GL}_n(F_v) \rightarrow \mathbf{R}^\times, \quad h_v \mapsto \frac{\det(h_v)}{|\det(h_v)|_v}.$$

By abuse of notation we also consider sgn_v and the norm $|\cdot|_v$ (implicitly composed with the determinant) as a character of $G_n(\mathbf{R})$. Recall that F is totally real. Then on real points we have the sign character

$$\text{sgn}_\infty := \otimes_v \text{sgn}_v : G_n(\mathbf{R}) \rightarrow \mathbf{R}^\times,$$

and the archimedean norm character

$$|\cdot|_\infty := \otimes_v |\cdot|_v : G_1(\mathbf{R}) \rightarrow \mathbf{R}^\times.$$

We obtain that for all $h \in G_n(\mathbf{R})$

$$\mathcal{N}(h) = \text{sgn}_\infty(h) \cdot |h|_\infty. \quad (6)$$

We remark that the group $\text{Hom}(G_n(\mathbf{R}), \mathbf{C}^\times)$ of quasi-characters of $G_n(\mathbf{R})$ is, as a complex manifold, a disjoint union of

$$2^{r_F} = \#\pi_0(G_n(\mathbf{R}))$$

copies of \mathbf{C} . Each component corresponds uniquely to a finite order character of $G_n(\mathbf{R})$. Each such character is of the form

$$\text{sgn}_\infty^\delta := \otimes_v \text{sgn}_v^{\delta_v} : G_n(\mathbf{R}) \rightarrow \mathbf{C}^\times,$$

with

$$\delta = (\delta_v)_v \in \prod_v \{0, 1\}.$$

Then the component of sgn_∞^δ is parametrized by the charts

$$\mathbf{C} \ni s \mapsto \omega_s^\delta := \text{sgn}_\infty^\delta \otimes |\cdot|_\infty^s \in \text{Hom}(G_n(\mathbf{R}), \mathbf{C}^\times). \quad (7)$$

We call ω_s^δ *algebraic* whenever $s \in \mathbf{Z}$. If χ is a quasi-character of $G_n(\mathbf{R})$, we set for $k \in \mathbf{Z}$

$$\chi[k] := \chi \otimes (\mathcal{N}^{\otimes k}).$$

If χ is algebraic, then so is $\chi[k]$.

We fix a non-trivial continuous character $\psi : F \backslash \mathbf{A}_F \rightarrow \mathbf{C}^\times$, and all Gauß sums we consider are understood with respect to ψ , and normalized in such a way that when $\chi = \chi' \otimes |\cdot|_{\mathbf{A}_F}^{k(\chi)}$, with χ' of finite order, then

$$G(\chi) := G(\chi') := \sum_{x \bmod f_{\chi'}} \chi' \left(\frac{x}{f_{\chi'}} \right) \psi \left(\frac{x}{f_{\chi'}} \right),$$

where $f_{\chi'} = \mathcal{O}_F \cdot f_{\chi'}$ is the conductor of the finite order character $\chi' : F^\times \backslash \mathbf{A}_F^\times$.

1.2 Rational models of compact groups

We fix for each m a \mathbf{Q} -form $K_m \subseteq G_m$ of the standard maximal compact subgroup of $G_m(\mathbf{R})$, i.e.

$$K_m(\mathbf{R}) = \prod_v \mathrm{O}(m, F_v) \quad (8)$$

where v runs through the respective archimedean places of F .

Corresponding to K_m we have a Cartan involution $\theta_m : G_m(\mathbf{R}) \rightarrow G_m(\mathbf{R})$, which is defined over \mathbf{Q} . Therefore we may and do consider θ_m as an automorphism of G_m over \mathbf{Q} . As we will see below, we may assume that K_m is split over an imaginary quadratic extension E_K/\mathbf{Q} and we may even choose $E_K = \mathbf{Q}(\sqrt{-1})$. We write $\tau_K \in \mathrm{Gal}(E_K/\mathbf{Q})$ for the unique non-trivial automorphism, which we simply call *complex conjugation*.

Following section 6.4 of [26] we fix the following admissible model of K_m (in the terminology of section 6 of loc. cit.), satisfying the above conditions having the additional property of being quasi-split at all finite primes $\neq 2$. Since F is totally real, we may consider the standard Cartan involution

$$\theta_m : \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_m \rightarrow \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_m, \quad g \mapsto g^{-t},$$

where the superscript $-t$ denotes matrix inversion composed with transposition. Then θ_m is defined over \mathbf{Q} and induces the standard Cartan involution on $\mathrm{GL}_m(F \otimes_{\mathbf{Q}} \mathbf{R})$, and thus gives rise to a \mathbf{Q} -rational model K_m of the standard maximal compact subgroup of the latter.

Let us show that K_m splits over $\mathbf{Q}(\sqrt{-1})$. We have a natural identification

$$F(\sqrt{-1}) = F \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{-1}).$$

It suffices to specify a Borel subgroup in

$$K_m(\mathbf{Q}(\sqrt{-1})) \subseteq (\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_m)(\mathbf{Q}(\sqrt{-1})) = \mathrm{GL}_m(F(\sqrt{-1})),$$

or equivalently a θ_m -stable Borel subalgebra in

$$\mathfrak{g}_{m, \mathbf{Q}(\sqrt{-1})} = \mathrm{Lie}(\mathrm{GL}_m(F(\sqrt{-1}))) = F(\sqrt{-1})^{m \times m}.$$

As Cartan subalgebra in \mathfrak{g}_{m, E_K} we choose inductively for $m \geq 3$

$$\mathfrak{h}_m := \mathfrak{h}_{m-2} \times \mathfrak{h}_2,$$

where in the low rank cases we choose

$$\mathfrak{h}_1 := F(\sqrt{-1})$$

and

$$\mathfrak{h}_2 := \left\{ \begin{pmatrix} a & \sqrt{-1}b \\ -\sqrt{-1}b & a \end{pmatrix} \mid a, b \in F(\sqrt{-1}). \right\}$$

Then the standard root space decomposition for \mathfrak{so}_m (see for example [28, Examples 2 and 4 in Chapter II.1]) shows that $\mathfrak{h}_m \cap \mathfrak{k}_{m, \mathbf{Q}(\sqrt{-1})}$ is a Cartan subalgebra of $\mathfrak{k}_{m, \mathbf{Q}(\sqrt{-1})}$ and the root space decomposition with respect to this Cartan subalgebra is defined over $\mathbf{Q}(\sqrt{-1})$.

We remark that we have a natural isomorphism

$$\pi_0(K_m) = \pi_0(K_m(\mathbf{R})),$$

where $K_m(\mathbf{R})$ on the right hand side is considered as a real Lie group. In particular we have

$$\pi_0(K_n) = \pi_0(K_n(\mathbf{R})) = \{\pm 1\}^{r_F}.$$

The standard inclusion $G_n \rightarrow G_{n+k}$, $k, n \geq 1$, induces a natural isomorphism

$$\pi_0(K_n) \cong \pi_0(K_{n+k}).$$

1.3 Rational models of pairs

We let G denote a connected reductive group over \mathbf{Q} , $K \subseteq G$ a closed reductive subgroup, and write $\mathfrak{g}, \mathfrak{k}, \mathfrak{g}_n, \dots$ for the Lie algebras of G, K, G_n, \dots respectively. All these Lie algebras are defined over \mathbf{Q} . To stress the base field E/\mathbf{Q} under consideration, we write

$$\mathfrak{g}_E := \mathfrak{g} \otimes_{\mathbf{Q}} E,$$

and similarly

$$G_E := G \times_{\mathbf{Q}} E$$

for the base change of $G \rightarrow \operatorname{Spec} \mathbf{Q}$ to $G_E \rightarrow \operatorname{Spec} E$. Then (\mathfrak{g}_E, K_E) is a reductive pair over E in the sense of section 1.4 of [26]. For the sake of readability we introduce the notation

$$(\mathfrak{g}, K)_E := (\mathfrak{g}_E, K_E),$$

that we also apply to more general pairs, which we will discuss in more detail in section 1.5 below.

We have a natural isomorphism between the group of quasi-characters $\operatorname{Hom}(G_n(\mathbf{R}), \mathbf{C}^\times)$ and the group of one-dimensional $(\mathfrak{g}_n, K_n)_{\mathbf{C}}$ -modules, sending a quasi-character χ to its derivative

$$d\chi : \mathfrak{g}_{n, \mathbf{C}} \rightarrow \mathbf{C},$$

and to the complexification of its restriction to $K_n(\mathbf{R})$,

$$\chi|_{K_n(\mathbf{R}), \mathbf{C}} : K_n(\mathbf{C}) \rightarrow \mathbf{C}^\times.$$

For the sake of notational simplicity we write ω_s^δ also for the image of ω_s^δ under this correspondence. Then, since \mathcal{N} is defined over \mathbf{Q} , and sgn_∞^δ , as a rational character of K_n , is defined over \mathbf{Q} as well, we see with (6), that for each $s \in \mathbf{Z}$ the algebraic quasi-character

$$\omega_s^\delta = \text{sgn}_\infty^\delta \otimes (\text{sgn}_\infty \otimes \eta)^s,$$

considered as a one-dimensional (\mathfrak{g}_n, K_n) -module, is defined over \mathbf{Q} as well. We write $X^{\text{alg}}(G_n(\mathbf{R}))$ for the group of algebraic quasi-characters of $G_n(\mathbf{R})$ or (\mathfrak{g}_n, K_n) , equivalently.

1.4 Finite-dimensional representations

Since G is quasi-split we may find a Borel $P \subseteq G$ defined over \mathbf{Q} . We choose a maximal torus $T \subseteq P$ over \mathbf{Q} and denote by P^- the opposite of P . We let $X(T)$ denote the characters of T defined over $\overline{\mathbf{Q}}$ or \mathbf{C} , which amounts to the same, and let $X_E(T)$ denote the subgroup of characters defined over E . Then the absolute Galois group $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and $\text{Aut}(\mathbf{C}/\mathbf{Q})$ act naturally on $X(T)$ from the right via the rule

$$\mu^\tau(t) = \mu(t^{\tau^{-1}})^\tau, \quad (9)$$

for $\mu \in X(T)$, $t \in T(\mathbf{C})$ and $\tau \in \text{Aut}(\mathbf{C}/\mathbf{Q})$. In general there is a second action on $X(T)$, introduced Borel and Tits in [5, 6], denoted $\Delta\tau(\mu)$ in loc. cit. The latter action maps dominant weights to dominant weights and reflects the action of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and $\text{Aut}(\mathbf{C}/\mathbf{Q})$ on the category of rational representations. However, since G is quasi-split these two actions agree in our case. In particular if M_μ denotes the irreducible rational G -module of highest weight $\mu \in X(T)$ over \mathbf{C} (or $\overline{\mathbf{Q}}$), then the Galois twisted module

$$(M_\mu)^\tau := M_\mu \otimes_{\mathbf{C}, \tau^{-1}} \mathbf{C} \quad (10)$$

is irreducible of highest weight μ^τ . Furthermore it is easy to see that M_μ is defined over the field of rationality $\mathbf{Q}(\mu)$ of μ . For all these assertions we refer to section 12 in [6].

1.5 Harish-Chandra modules

If (V, ρ) is a Casselman-Wallach representation of $G(\mathbf{R})$, we write $V^{(K)}$ for the subspace of $K(\mathbf{R})$ -finite vectors. This is a finitely generated admissible $(\mathfrak{g}, K)_\mathbf{C}$ -module, and is irreducible if and only if V is (topologically) irreducible. Indeed, the categories of finite length $(\mathfrak{g}, K)_\mathbf{C}$ -modules is equivalent to the category of Casselman-Wallach representations of $G(\mathbf{R})$, i.e. finitely generated, admissible, Fréchet representations of moderate growth.

We give a sketch of the theory of Harish-Chandra modules over arbitrary fields of characteristic 0. The relevant theory of cohomological induction over general fields will be taken up in section 2. For the fundamental theory of modules and cohomological induction over arbitrary fields, as well as the rationality results we need, we refer to [26].

A pair over a field E/\mathbf{Q} consists of an E -Lie algebra \mathfrak{a}_E and a reductive algebraic group B_E over E , together with an inclusion

$$\iota_E : \operatorname{Lie}(B_E) \rightarrow \mathfrak{a}_E,$$

of Lie algebras and an action of B_E on \mathfrak{a}_E , extending the action of B_E on $\operatorname{Lie}(B_E)$, whose derivative is the adjoint action of $\operatorname{Lie}(B_E)$ on \mathfrak{a}_E , the latter action being induced by ι_E .

Then an $(\mathfrak{a}, B)_E$ -module X_E consists of an E -vector space X_E together with compatible actions of \mathfrak{a}_E and B_E . Here we implicitly assume that X_E is a *rational* B_E -module, which amounts to saying that it is a direct sum of finite-dimensional rational representations of B_E . Then this rational action induces an action of $\operatorname{Lie}(B_E)$ on X_E , and we assume the action of \mathfrak{a}_E to be an extension of this action. Furthermore we assume the given adjoint action of B_E on \mathfrak{a}_E and the given action on X_E to be compatible with the action of \mathfrak{a}_E in the usual sense.

Then the category $\mathcal{C}(\mathfrak{a}, B)_E$ of $(\mathfrak{a}, B)_E$ -modules (over E) is an abelian category, even an E -linear tensor category. For each extension of fields E'/E we have natural base change functors

$$- \otimes_E E' : \mathcal{C}(\mathfrak{a}, B)_E \rightarrow \mathcal{C}(\mathfrak{a}, B)_{E'},$$

sending an E -rational module X_E to the E' -rational module

$$X_{E'} := X_E \otimes_E E',$$

which is an $(\mathfrak{a}, B)_{E'}$ -module. A fundamental property of the base change functor is

$$\operatorname{Hom}_{\mathfrak{a}, B}(X_E, Y_E) \otimes_E E' = \operatorname{Hom}_{\mathfrak{a}, B}(X_{E'}, Y_{E'})$$

for all $X_E, Y_E \in \mathcal{C}(\mathfrak{a}, B)_E$, cf. Proposition 1.1 in [26]. This observation enables us to control the rationality of functionals.

In this context we remark that the algebraic quasi-characters in $X^{\operatorname{alg}}(G_n(\mathbf{R}))$, since characters of the pair (\mathfrak{g}_n, K_n) , are all defined over \mathbf{Q} , another observation of fundamental importance.

We remark furthermore that if (\mathfrak{a}, B) is a pair over \mathbf{Q} , then the Galois action on modules defined in (10) extends naturally to arbitrary (\mathfrak{a}, B) -modules.

2 Cohomologically induced modules over \mathbf{Q}

In this section we discuss the fields of definition of tensor products of tempered cohomologically induced standard modules on products of GL_n . We first recall classical results which are fundamental to our subsequent treatment.

2.1 Representations of orthogonal groups

For $n \geq 1$, we choose a maximal torus $T \subseteq \mathrm{SO}(2n+1)$. Then the root system $\Delta \subseteq X^*(T)$ of $\mathrm{SO}(2n+1)$ is of type B_n , and we identify

$$X^*(T) \otimes_{\mathbf{Z}} \mathbf{R} = \mathbf{R}^n,$$

by means of the standard orthonormal basis e_1, \dots, e_n of \mathbf{R}^n . We assume that in our notation the root system is given by

$$\Delta = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\},$$

our choice of simple roots in this notation being

$$e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n.$$

The orthogonal group $\mathrm{O}(2n+1)$ only has inner automorphisms and is a direct product

$$\mathrm{O}(2n+1) = \mathrm{SO}(2n+1) \times \{\pm 1\},$$

and thus irreducible $\mathrm{O}(2n+1)$ -modules $W^\pm(\lambda)$ are indexed by a sign \pm and analytically integral dominant weights

$$\lambda = \lambda_1 e_1 + \dots + \lambda_n e_n, \quad \lambda_1, \dots, \lambda_n \in \mathbf{Z},$$

for $\mathrm{SO}(2n+1)$, satisfying the dominance condition

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0.$$

In the even case we consider for $n \geq 2$ the root system

$$\Delta \subseteq X^*(T) \subseteq X^*(T) \otimes_{\mathbf{Z}} \mathbf{R} = \mathbf{R}^n,$$

of $\mathrm{SO}(2n)$ for a maximal torus $T \subseteq \mathrm{SO}(2n)$, with the same standard basis as above. It is of type D_n , and we have

$$\Delta = \{\pm e_i \pm e_j \mid i < j\}.$$

We fix the simple roots as

$$e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_{n-1} + e_n.$$

The outer automorphism group of $\mathrm{O}(2n)$ is of order two and the orthogonal group $\mathrm{O}(2n)$ is the unique non-split semidirect product

$$\mathrm{O}(2n) = \mathrm{SO}(2n) \rtimes \{\pm 1\}.$$

By classical Mackey Theory there are two distinct cases for the structure of irreducible rational $\mathrm{O}(2n)$ -modules. Given an analytically integral dominant weight

$$\lambda = \lambda_1 e_1 + \dots + \lambda_n e_n, \quad \lambda_1, \dots, \lambda_n \in \mathbf{Z},$$

of $\mathrm{SO}(2n)$, i.e. satisfying

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n|,$$

the induced representation

$$W(\lambda) = \mathrm{Ind}_{\mathrm{SO}(2n)}^{\mathrm{O}(2n)} V(\lambda)$$

is irreducible whenever $\lambda_n \neq 0$, and its isomorphism class depends only on the tuple $(\lambda_1, \lambda_2, \dots, |\lambda_n|)$. In the case $\lambda_n = 0$, the module $\mathrm{Ind}_{\mathrm{SO}(2n)}^{\mathrm{O}(2n)} V(\lambda)$ decomposes into two non-isomorphic irreducible $\mathrm{O}(2n)$ -representations $W^\pm(\lambda)$, which differ by a sign twist.

The following classical result will be relevant for our descent argument in the proof of Theorem 2.3.

Proposition 2.1. *For every $n \geq 1$ and every analytically integral dominant weight λ for $\mathrm{SO}(2n+1)$ and every sign \pm , the irreducible complex $\mathrm{O}(2n+1)$ -module $W^\pm(\lambda)$ is self-dual and real, i.e. defined over \mathbf{R} , hence so is the underlying irreducible $\mathrm{SO}(2n+1)$ -module.*

Likewise, for every analytically integral dominant weight λ for $\mathrm{SO}(2n)$ with $\lambda_n \neq 0$ the irreducible complex $\mathrm{O}(2n)$ -module $W(\lambda)$ is always real. As a complex $\mathrm{SO}(2n)$ -module it decomposes into a direct sum

$$W(\lambda) = V(\lambda) \oplus V(\tilde{\lambda})$$

of two irreducible complex $\mathrm{SO}(2n)$ -modules. $V(\lambda)$ (resp. $V(\tilde{\lambda})$) is real if and only if it is self-dual. This is so if and only if n is even.

Proof. By Proposition 6.10 in [26] every self-dual irreducible complex $\mathrm{SO}(2n)$ - and $\mathrm{SO}(2n+1)$ -module is real. Hence the claim for $\mathrm{O}(2n+1)$ and $\mathrm{O}(2n)$ follows with Lemma 6.3 of loc. cit.

The only detail which is not explicitly covered by this reasoning is the last statement that for a dominant weight λ with $\lambda_n \neq 0$ the module $V(\lambda)$ is self-dual only for even n . If n is even the action of $-w_0$ on the weight space is trivial. For odd n we have for the weight λ under consideration

$$-w_0(\lambda) = (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n) = \tilde{\lambda},$$

and since $\lambda_n \neq 0$, the claim follows. \square

2.2 Induction data

We fix a finite sequence n_1, n_2, \dots, n_r of $r \geq 1$ positive integers and set

$$G := \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_r},$$

where F/\mathbf{Q} is any totally real number field. As before we write \mathfrak{g} for the rational Lie algebra of G , as well as the standard choice

$$K := \mathrm{Res}_{F/\mathbf{Q}} \mathrm{O}(n_1) \times \mathrm{O}(n_2) \times \cdots \times \mathrm{O}(n_r) \subseteq G$$

of a rational model of a standard maximal compact subgroup of $G(\mathbf{R})$ as in section 1.2. Along with K comes a Cartan involution θ of G and \mathfrak{g} , which is defined over \mathbf{Q} as well.

The product decomposition of G induces a product decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r, \quad (11)$$

and similarly for the Lie algebra \mathfrak{k} of K .

We recall that we fixed an imaginary quadratic extension E_K/\mathbf{Q} where K is split. In later parts of the text we will assume that $E_K = \mathbf{Q}(\sqrt{-1})$, but in this section a general E_K is just as good. We fix a θ -stable Borel subalgebra $\mathfrak{q} \subseteq \mathfrak{g}_{E_K}$, where θ -stability means that

$$\theta(\mathfrak{q}) = \mathfrak{q}. \quad (12)$$

Later in the text we will assume \mathfrak{q} to be transversal to a specific diagonally embedded Lie algebra \mathfrak{h} , but for now any θ -stable \mathfrak{q} which factors as a product

$$\mathfrak{q} = \mathfrak{q}_1 \times \mathfrak{q}_2 \times \cdots \times \mathfrak{q}_r,$$

according to (11) is sufficient for our purpose.

By (12), the Lie algebra

$$\mathfrak{q} \cap \mathfrak{k}_{E_K} \subseteq \mathfrak{k}_{E_K}$$

is a Borel subalgebra of \mathfrak{k}_{E_K} . Again by (12), complex conjugation $\tau_K \in \text{Gal}(E_K/\mathbf{Q})$ maps \mathfrak{q} to its opposite

$$\overline{\mathfrak{q}} := \mathfrak{q}^{\tau_K} = \mathfrak{q}^-,$$

if we consider the unique θ -stable and \mathbf{Q} -rational Levi factor given explicitly by

$$\mathfrak{c} := \mathfrak{q} \cap \overline{\mathfrak{q}}.$$

Again \mathfrak{c} factors as a product

$$\mathfrak{c} = \mathfrak{c}_1 \times \mathfrak{c}_2 \times \cdots \times \mathfrak{c}_r,$$

accordingly. Let $\mathfrak{u} \subseteq \mathfrak{q}$ denote the nilpotent radical of \mathfrak{q} . Then $\mathfrak{q}_{E_K} \cap \mathfrak{k}_{E_K}$ has the Levi decomposition

$$\mathfrak{q}_{E_K} \cap \mathfrak{k}_{E_K} = (\mathfrak{c}_{E_K} \cap \mathfrak{k}_{E_K}) \oplus (\mathfrak{u}_{E_K} \cap \mathfrak{k}_{E_K}).$$

We fix the compact factor C of the Levi pair associated to \mathfrak{q} as follows. For each index $1 \leq i \leq r$ we set

$$C_i := Z_{K_i}(\mathfrak{c}_i).$$

If n is even, then C_i is contained in K_i^0 ; if n is odd, it meets every connected component of K_i . Finally

$$C := C_1 \times C_2 \times \cdots \times C_r.$$

Then C is defined over \mathbf{Q} and $(\mathfrak{q}, C)_{E_K}$ is what we call a *parabolic pair* over E_K .

We consider rational models of cohomologically induced $(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}})$ -standard-modules $A_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ arising as follows.

Starting from a rational absolutely irreducible representation $M(\mu)$ of G with highest weight μ with respect to $\mathfrak{q}_{\mathbf{C}}$ we have a one-dimensional highest weight space $H^0(\mathfrak{u}; M(\mu))$ inside of $M(\mu)$. Since C normalizes \mathfrak{q} , the inclusion $C \rightarrow G$ induces an action of C on $M(\mu)$ under which $H^0(\mathfrak{u}; M(\mu))$ is stable. Hence we obtain a character μ of the pair $(\mathfrak{c}, C)_{\mathbf{C}}$, or equivalently of the parabolic pair $(\mathfrak{q}, C)_{\mathbf{C}}$ with trivial action of the radical. This agrees with the natural action of the pair $(\mathfrak{q}, C)_{\mathbf{C}}$ on $H^0(\mathfrak{u}; M(\mu))$. All these data are indeed defined over $E_K(\mu) = E_K \cdot \mathbf{Q}(\mu)$, where $\mathbf{Q}(\mu)$ is the field of rationality of $M(\mu)$, once we fix an inclusion

$$\iota : E_K(\mu) \rightarrow \mathbf{C}. \quad (13)$$

We assume this to be the case in all what follows.

In [26] the author introduced Zuckerman functors over arbitrary fields of characteristic 0. We set

$$S_{\mathfrak{q}} := \dim(\mathfrak{u}_{E_K} \cap \mathfrak{k}_{E_K}),$$

and denote by $R^q \Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, K}$ the $E_K(\mu)$ -rational q -th right derived Zuckerman functor for the inclusion of pairs $(\mathfrak{g}, C) \rightarrow (\mathfrak{g}, K)$, as introduced in [26]. The construction of Zuckerman functors in loc. cit. generalizes Zuckerman's original construction discussed in [48]. For the fundamental properties of Zuckerman's cohomological induction over \mathbf{C} we refer to the monograph [29].

As in the classical case, the functor $\Gamma_{\mathfrak{g}, CK^0}^{\mathfrak{g}, K}$ is exact and given by induction along the inclusion $CK^0 \rightarrow K$. Now $\Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, CK^0}$, being right adjoint to an exact forgetful functor, maps injectives to injectives. Therefore the Grothendieck spectral sequence associated to the composition

$$\Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, K} = \Gamma_{\mathfrak{g}, CK^0}^{\mathfrak{g}, K} \circ \Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, CK^0}$$

induces in each degree q a natural isomorphism of functors

$$R^q \Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, K} = \Gamma_{\mathfrak{g}, CK^0}^{\mathfrak{g}, K} \circ R^q \Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, CK^0} \quad (14)$$

Using the author's rational construction, we have the $E_K(\mu)$ -rational module

$$A_{\mathfrak{q}}(\mu) := R^{S_{\mathfrak{q}}} \Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, K}(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}_{E_K(\mu)}), \mu \otimes \bigwedge^{\dim \mathfrak{u}} \mathfrak{u}_{E_K(\mu)}^{(C)})),$$

where the superscript $(\cdot)^{(C)}$ denotes the subspace of C -finite vectors as before. Then, since rational Zuckerman functors commute with base change (Theorem 2.5 of [26]), we have an isomorphism

$$A_{\mathfrak{q}}(\mu) \otimes \mathbf{C} \cong A_{\mathfrak{q}}(\mu)_{\mathbf{C}},$$

which we use to identify the right hand side with the left hand side, justifying this abuse of notation.

For the study of periods we introduce the intermediate module

$$A_{\mathfrak{q}}^{\circ}(\mu) := R^{S_{\mathfrak{q}}} \Gamma_{\mathfrak{g}, C}^{\mathfrak{g}, CK^0} (\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}_{E_K(\mu)}), \mu \otimes \bigwedge^{\dim \mathfrak{u}} \mathfrak{u}_{E_K(\mu)})^{(C)}).$$

By (14) its relation to $A_{\mathfrak{q}}(\mu)$ is

$$A_{\mathfrak{q}}(\mu)_{E_K} = \Gamma_{\mathfrak{g}, CK^0}^{\mathfrak{g}, K}(A_{\mathfrak{q}}^{\circ}(\mu)_{E_K}), \quad (15)$$

and by Frobenius reciprocity we may consider $A_{\mathfrak{q}}^{\circ}(\mu)$ as an irreducible $(\mathfrak{g}, CK^0)_{E_K(\mu)}$ -submodule of $A_{\mathfrak{q}}(\mu)_{E_K(\mu)}$, and the K -translates of the former generate the latter as an $E_K(\mu)$ -vector space.

The author showed in Theorem 7.3 of [26] that $A_{\mathfrak{q}}(\mu)$ is actually defined over $\mathbf{Q}(\mu)$. In general this is not true for $A_{\mathfrak{q}}^{\circ}(\mu)$. We will determine its field of definition in Theorem 2.3 below. This result is crucial for our applications to special values.

For the sake of completeness, we remark that the natural embedding $\mathbf{Q}(\mu) \rightarrow E_K(\mu)$ induces with (13) an embedding $\mathbf{Q}(\mu) \rightarrow \mathbf{C}$, which we understand fixed in the sequel. In particular (31) retains its meaning also for the model of $A_{\mathfrak{q}}(\mu)$ over $\mathbf{Q}(\mu)$, the latter being unique by virtue of Proposition 3.4 of loc. cit.

2.3 The bottom layer

In this section we recall fundamental facts about the structure of bottom layer of the standard modules of interest as a rational representation of K and K^0 , respectively. The understanding of rationality properties of the bottom layer is the key to the understanding of the rationality properties of the intermediate modules $A_{\mathfrak{q}}^{\circ}(\mu)$.

We denote by

$$B_{\mathfrak{q}}(\mu)_{E_K(\mu)} := A_{\mathfrak{q} \cap \mathfrak{k}}(\mu|_C \otimes \bigwedge^{\dim \mathfrak{u}/\mathfrak{u} \cap \mathfrak{k}} (\mathfrak{u}/\mathfrak{u} \cap \mathfrak{k}))_{E_K(\mu)} \subseteq A_{\mathfrak{q}}(\mu)_{E_K(\mu)}$$

the bottom layer, with

$$A_{\mathfrak{q} \cap \mathfrak{k}}(\cdot) := R^{S_{\mathfrak{q}}} \Gamma_{\mathfrak{k}, C}^{\mathfrak{k}, K} \left((\cdot) \otimes \bigwedge^{\dim \mathfrak{u} \cap \mathfrak{k}} \mathfrak{u} \cap \mathfrak{k} \right).$$

The descent argument in the proof of Theorem 7.3 in [26] tells us that the bottom layer itself is defined over $\mathbf{Q}(\mu)$. Again its model over $\mathbf{Q}(\mu)$ is unique and we denote it the same.

As in the non-compact case we introduce the intermediate module

$$B_{\mathfrak{q}}^{\circ}(\mu|_C)_{E_K(\mu)} := A_{\mathfrak{q} \cap \mathfrak{k}}^{\circ}(\mu|_C)_{E_K(\mu)} := R^{S_{\mathfrak{q}}} \Gamma_{\mathfrak{k}, C}^{\mathfrak{k}, CK^0} (\mu|_C \otimes \bigwedge^{\dim \mathfrak{u}} \mathfrak{u}_{E_K(\mu)}) \subseteq A_{\mathfrak{q}}^{\circ}(\mu)_{E_K(\mu)}.$$

This is an irreducible representation of K^0 of highest weight

$$\mu_K := \mu|_{C^0} + 2\rho(\mathfrak{u}/\mathfrak{u} \cap \mathfrak{k}), \quad (16)$$

where

$$\rho(\mathfrak{u}/\mathfrak{u} \cap \mathfrak{k}) = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u}/\mathfrak{u} \cap \mathfrak{k}, \mathfrak{c})} \alpha,$$

is the half sum of weights in $\mathfrak{u}/\mathfrak{u} \cap \mathfrak{k}$. Again we have

$$B_{\mathfrak{q}}(\mu)_{E_K(\mu)} = \Gamma_{\mathfrak{g}, CK^0}^{\mathfrak{g}, K}(B_{\mathfrak{q}}^{\circ}(\mu)_{E_K(\mu)}). \quad (17)$$

2.4 Structure of the bottom layer

We assume that n_1, \dots, n_s are even and n_{s+1}, \dots, n_r are odd, for some $0 \leq s \leq r$. By the preceding discussion and the transitivity principle (17), the representation $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ decomposes as a complex representation of

$$K^0(\mathbf{R}) = \prod_{v|\infty} \mathrm{SO}(n_1, F_v) \times \mathrm{SO}(n_2, F_v) \times \cdots \times \mathrm{SO}(n_r, F_v),$$

into a direct sum

$$B_{\mathfrak{q}}(\mu)_{\mathbf{C}} = \bigotimes_{v|\infty} (B_{\mu_{v,1}} \oplus B_{\tilde{\mu}_{v,1}}) \otimes \cdots \otimes (B_{\mu_{v,s}} \oplus B_{\tilde{\mu}_{v,s}}) \otimes B_{\mu_{v,s+1}} \otimes \cdots \otimes B_{\mu_{v,r}}. \quad (18)$$

where $B_{\mu_{v,i}}$ is the complex irreducible representation of $\mathrm{SO}(n_i, F_v)$ of highest weight $\mu_{v,i}$, and

$$\mu_K = \otimes_v (\otimes_i \mu_{v,i})$$

is the natural factorization of μ_K into local components. Similarly we have

$$B_{\mathfrak{q}}^{\circ}(\mu)_{\mathbf{C}} = \bigotimes_v B_{\mu_{v,1}} \otimes \cdots \otimes B_{\mu_{v,s}}, \quad (19)$$

again as a complex representation of $K^0(\mathbf{R})$.

We remark that since the representation $B_{\mathfrak{q}}^{\circ}(\mu)_{\mathbf{C}}$ is defined over $E_K(\mu)$, the summands in the direct sum decompositions (3) and (18) are defined over $E_K(\mu)$ as well. However, we need to determine their field of definition.

Lemma 2.2. *The representation $B_{\mathfrak{q}}^{\circ}(0)_{\mathbf{C}}$ of K^0 and the (\mathfrak{g}, K^0) -module $A_{\mathfrak{q}}^{\circ}(0)_{\mathbf{C}}$ are defined over E_K . The module $B_{\mathfrak{q}}^{\circ}(0)_{\mathbf{C}}$ resp. $A_{\mathfrak{q}}^{\circ}(0)_{\mathbf{C}}$ is defined over \mathbf{Q} if and only if for every $1 \leq i \leq r$ we have*

$$2 \mid n_i \implies 4 \mid n_i. \quad (20)$$

Proof. By the descent argument (cf. proof of Theorem 7.3 in [26]), the statement of the lemma about the bottom layer is equivalent to the statement that the complex representation $B_{\mathfrak{q}}^{\circ}(0)_{\mathbf{C}}$ is real if and only if (20) is satisfied.

For $B_q^\circ(0)_\mathbb{C}$ to be real it is necessary that $B_q^\circ(0)_\mathbb{C}$ be self-dual. By Proposition 2.1 this cannot be the case if there is an index $1 \leq i \leq r$ for which $n_i \equiv 2 \pmod{4}$. Moreover self-duality is also a sufficient condition, again by Proposition 2.1. Hence the claim for $B_q^\circ(0)_\mathbb{C}$ follows.

As in the proof of Theorem 7.3 of [26] we may appeal to Proposition 5.8 of loc. cit. to conclude that $A_q^\circ(0)_\mathbb{C}$ is defined over \mathbf{Q} . Alternatively we may argue that, as a submodule

$$A_q^\circ(0)_{E_K} \subseteq A_q(0)_{E_K}$$

of a module which is defined over \mathbf{Q} (Theorem 7.3 of loc. cit.), it is defined over \mathbf{Q} if and only if it is invariant under the action of $\text{Gal}(E_K/\mathbf{Q})$, and this is so if and only if $B_q^\circ(0)_{E_K}$ is invariant under the action of $\text{Gal}(E_K/\mathbf{Q})$, since each irreducible submodule of $A_q(0)_{E_K}$ is uniquely determined by the unique minimal K^0 -type it contains. \square

Theorem 2.3. *The K^0 -module $B_q^\circ(\lambda)_\mathbb{C}$ and the (\mathfrak{g}, K^0) -module $A_q^\circ(\lambda)_\mathbb{C}$ are defined over $E_K(\mu)$. If $\sqrt{-1} \notin \mathbf{Q}(\mu)$, then $B_q^\circ(\lambda)_\mathbb{C}$ resp. $A_q^\circ(\lambda)_\mathbb{C}$ is defined over $\mathbf{Q}(\mu)$ if and only if for every index $1 \leq i \leq r$,*

$$2 \mid n_i \implies 4 \mid n_i.$$

We remark that the same rationality statement applies to the summands in the direct sum decomposition (18) and in the decomposition of $A_q(\mu)$ into absolutely irreducible (\mathfrak{g}, K^0) -modules.

Proof. We follow the argument of the proof of Proposition 7.1 in [26]. By Lemma 2.2 we know that the $(\mathfrak{g}, K^0)_{E_K}$ -submodule

$$A_q^\circ(0)_{E_K} \subseteq A_q(0)_{E_K}$$

is defined over \mathbf{Q} if and only if (20) is satisfied for all $1 \leq i \leq r$. Assume $\sqrt{-1} \notin \mathbf{Q}(\mu)$ and consider for each dominant integral weight ν the translation functor

$$\mathcal{T}_\mu^\nu : X \mapsto \pi_\nu(X \otimes M(\mu)),$$

where π_ν denotes the projection onto the submodule on which the center $Z(\mathfrak{g}) \subseteq U(\mathfrak{g})$ acts as in $M(\nu)$. Then we have a commutative square

$$\begin{array}{ccc} A_q^\circ(\mu)_{E_K(\mu)} & \xrightarrow{\sim} & \mathcal{T}_\mu^\mu(A_q^\circ(0)_{E_K(\mu)}) \\ \downarrow & & \downarrow \\ A_q(\mu)_{E_K(\mu)} & \xrightarrow{\sim} & \mathcal{T}_\mu^\mu(A_q(0)_{E_K(\mu)}) \end{array}$$

and likewise for the contragredient weight μ^\vee ,

$$\begin{array}{ccc} A_q^\circ(0)_{E_K(\mu)} & \xrightarrow{\sim} & \mathcal{T}_{\mu^\vee}^0(A_q^\circ(\mu)_{E_K(\mu)}) \\ \downarrow & & \downarrow \\ A_q(0)_{E_K(\mu)} & \xrightarrow{\sim} & \mathcal{T}_{\mu^\vee}^0(A_q(\mu)_{E_K(\mu)}) \end{array}$$

Therefore, $A_q^\circ(\mu)_{E_K(\mu)}$ is defined over $\mathbf{Q}(\mu)$ if and only if $A_q^\circ(0)_{E_K(\mu)}$ has a model over $\mathbf{Q}(\mu)$.

In light of Lemma 2.2, this proves the implication

$$(\forall i : 2 \mid n_i \implies 4 \mid n_i) \implies A_q^\circ(\mu)_{E_K(\mu)} \text{ is defined over } \mathbf{Q}(\mu).$$

To see the other implication, it remains to show by the preceding discussion that if $A_q^\circ(0)_{E_K(\mu)}$, being a translate of $A_q^\circ(\mu)_{E_K(\mu)}$, has a model over $\mathbf{Q}(\mu)$, then $A_q^\circ(0)_{E_K}$ has already a model over \mathbf{Q} .

Due to our assumption that $\sqrt{-1} \notin \mathbf{Q}(\mu)$, the extensions E_K and $\mathbf{Q}(\mu)$ are linearly disjoint over \mathbf{Q} , and restriction induces an isomorphism of Galois groups

$$\mathrm{Gal}(E_K(\mu)/\mathbf{Q}(\mu)) \xrightarrow{\sim} \mathrm{Gal}(E_K/\mathbf{Q}). \quad (21)$$

Therefore, if $A_q^\circ(0)_{E_K(\mu)}$ has a model over $\mathbf{Q}(\mu)$, complex conjugation $\tau_K \in \mathrm{Gal}(E_K(\mu)/\mathbf{Q}(\mu))$ stabilizes $A_q^\circ(0)_{E_K(\mu)}$ as a submodule of $A_q(0)_{E_K(\mu)}$. By (21) it therefore stabilizes the subspace $A_q^\circ(0)_{E_K}$, which, by Galois descent for vector spaces, therefore is defined over \mathbf{Q} . This implies, with Lemma 2.2,

$$\forall i : 2 \mid n_i \implies 4 \mid n_i. \quad (22)$$

This concludes the proof of the statement about standard modules.

To determine the fields of definition of the minimal K^0 -types, we observe that

$$B_q^\circ(\mu)_{E_K(\mu)} = B_q(\mu)_{E_K(\mu)} \cap A_q^\circ(\mu)_{E_K(\mu)},$$

which shows that $B_q^\circ(\mu)_{E_K(\mu)}$ is defined over $\mathbf{Q}(\mu)$ whenever (22) is satisfied.

Assume conversely that $B_q^\circ(\mu)_{E_K(\mu)}$ is defined over $\mathbf{Q}(\mu)$. As in the end of the proof of Lemma 2.2 we may argue via Galois descent for vector spaces, that this implies that $A_q^\circ(\mu)_{E_K(\mu)}$, as a submodule of $A_q(\mu)_{E_K(\mu)}$, is defined over $\mathbf{Q}(\mu)$, whence (22). \square

In the case $F = \mathbf{Q}$, $s = 1$, i.e. if there is only one even index n_1 , and if furthermore $4 \nmid n_1$, the decomposition (18) corresponds to the decomposition of $A_q(\mu)$ into the sum of a holomorphic and antiholomorphic discrete series representation. It also reflects the decomposition of (\mathfrak{g}, GK^0) -cohomology that we will discuss below.

3 Rankin-Selberg convolutions

From this section on and for all what follows, we put ourselves in the situation where $G = G_{n+1} \times G_n$ and $H = G_n \subseteq G$ is diagonally embedded via

$$h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}.$$

We fix the \mathbf{Q} -rational model of a maximal compact subgroup of G as $K := K_{n+1} \times K_n$ with K_n as in section 1.2 and let $L := H \cap K$. Then $L = K_n \subseteq G_n = H$.

Any pair of cuspidal automorphic representation π and σ of $\mathrm{GL}_{n+1}(\mathbf{A}_F)$ and $\mathrm{GL}_n(\mathbf{A}_F)$, respectively, gives rise to an automorphic representation Π of $G(\mathbf{A})$, which is the completed tensor product of π and σ . We call Π *regular algebraic* if π and σ are regular algebraic. In the same spirit we identify the local components Π_v , v a place of \mathbf{Q} , with the corresponding (completed) tensor products of the corresponding local components of π and σ . Representations Π_∞ , π_∞ , σ_∞ at ∞ are always assumed to be smooth, i.e. departing possibly from a Hilbert space representation, we implicitly pass to the subspace of smooth vectors with the corresponding Fréchet topology. Departing from an irreducible Π , the archimedean representations then are irreducible Casselman-Wallach representations, and Π_∞ is a completed projective tensor product $\pi_\infty \hat{\otimes} \sigma_\infty$.

We write $N = N_{n+1} \times N_n \subseteq G = G_{n+1} \times G_n$ for the restriction of scalars of the group of unipotent upper triangular matrices, i.e. $N_m \subseteq G_m$ denotes the restriction of scalars of the group of unipotent upper triangular matrices in GL_m . We fix a non-trivial continuous character

$$\psi : N(\mathbf{Q}) \backslash N(\mathbf{A}) \rightarrow \mathbf{C}^\times$$

with the property that its restriction to $N(\mathbf{Q}) \cap H(\mathbf{Q}) \backslash N(\mathbf{A}) \cap H(\mathbf{A})$ be trivial. To be more concrete, we assume ψ to be of the form

$$((u_{ij})_{1 \leq i, j \leq n+1} \times (v_{ij})_{1 \leq i, j \leq n} \mapsto \prod_{k=1}^n \psi(u_{kk+1}) \cdot \prod_{l=1}^{n-1} \psi(v_{ll+1})^{-1}).$$

We fix Haar measures dg , dn , dh on $G(\mathbf{A})$, $N(\mathbf{A})$ and $H(\mathbf{A})$, and use subscripts $(\cdot)_v$ to denote their local factors at a place v .

We denote by $\mathscr{W}(\Pi, \psi)$ the ψ -Whittaker model of Π , and we are in particular concerned with the model at infinity $\mathscr{W}(\Pi_\infty, \psi_\infty)$. Since Π is a cuspidal representation of $G(\mathbf{A})$, Π is globally generic, and hence the Whittaker model exists [41]. We are interested in the Rankin-Selberg L -function $L(s, \Pi)$ in the sense of Jacquet, Shalika and Piatetski-Shapiro [21].

3.1 The archimedean zeta integral

For each quasi-character $\chi \in \mathrm{Hom}(H(\mathbf{R}), \mathbf{C}^\times)$ and each $w \in \mathscr{W}(\Pi_\infty, \psi_\infty)$, the archimedean Rankin-Selberg integral

$$\Psi_\infty : (\chi, w) \mapsto \int_{N(\mathbf{R}) \backslash H(\mathbf{R})} w(h_\infty) \chi(h_\infty) dh_\infty \quad (23)$$

converges absolutely, provided χ lies a suitable right half plane in the sense of (7). In such a half plane $\Psi_\infty(-, w)$ defines a holomorphic function in χ . More concretely the map

$$e_\infty : (\chi, w) \mapsto \frac{\Psi_\infty(\chi, w)}{L(\frac{1}{2}, \Pi_\infty \otimes \chi)} \quad (24)$$

is well defined for *all* χ , since the right hand side defines an entire function in χ , cf. Theorem 1.2 in [10], $\mathrm{Hom}(H(\mathbf{R}), \mathbf{C}^\times)$ being a disjoint union of finitely many

copies of \mathbf{C} . For each χ we may find a $w \in \mathscr{W}(\Pi_\infty, \psi_\infty)^{(K)}$ with the property that

$$e_\infty(\chi, w) \neq 0. \quad (25)$$

Following an argument of Jacquet and Shalika (see the remark after Theorem 1.3 in [10]), we know that for fixed K -finite w , the map

$$\chi \mapsto e_\infty(\chi, w)$$

is on each connected component of $\text{Hom}(H(\mathbf{R}), \mathbf{C}^\times)$ given by a polynomial in the complex parameter of that component. This in particular implies that $e_\infty(-, w)$ is locally constant in the variable χ whenever (25) is satisfied for *all* quasi-characters χ . In other words, for such a w we know that the Rankin-Selberg integral (23) represents the Γ -factor of the Rankin-Selberg L -function $L(s, \Pi \otimes \chi)$ up to a complex unit, the latter only depending on the connected component containing χ . We will see in Theorem 4.8 that we may indeed choose a vector w which satisfies (25) for all quasi-characters χ .

In a representation theoretic sense the integral (23) for χ in a suitable right half plane and (24) for general χ , therefore defines, in the variable w , a continuous $H(\mathbf{R})$ -equivariant functional

$$e_\infty(\chi, -) : \mathscr{W}(\Pi_\infty, \psi_\infty) \rightarrow \chi_{\mathbf{C}}^{-1},$$

the right hand side denoting the one-dimensional complex $H(\mathbf{R})$ -module with action given by the inverse quasi-character χ^{-1} .

We remark that since $\Pi_\infty \otimes \chi \cong \Pi_\infty$ for every finite order character χ , we have for such a χ an identity of local L -functions

$$L(s, \Pi_\infty \otimes \chi) = L(s, \Pi_\infty).$$

3.2 The non-archimedean zeta integrals

Fix a rational prime p . For each quasi-character $\chi \in \text{Hom}(H(\mathbf{Q}_p), \mathbf{C}^\times)$ and each $w \in \mathscr{W}(\Pi_p, \psi_p)$, the non-archimedean analogue of (23) is

$$\Psi_p : (\chi, w) \mapsto \int_{N(\mathbf{Q}_p) \backslash H(\mathbf{Q}_p)} w(h_p) \chi(h_p) dh_p. \quad (26)$$

Again for χ in a suitable right half plane the integral in (26) is absolutely convergent and in such a half plane $\Psi_p(-, w)$ defines a holomorphic function in χ . The map

$$e_p : (\chi, w) \mapsto \frac{\Psi_p(\chi, w)}{L(\frac{1}{2}, \Pi_p \otimes \chi)} \quad (27)$$

is well defined for all χ and w and defines an entire function in the variable χ on each connected component of $\text{Hom}(H(\mathbf{Q}_p), \mathbf{C}^\times)$. As in the archimedean case we always find for a given χ a $w \in \mathscr{W}(\Pi_p, \psi_p)$ such that

$$e_p(\chi, w) \neq 0.$$

This then implies that for each quasi-character χ there is a *good vector* t_p^χ , depending only on the connected component of χ in $\text{Hom}(H(\mathbf{Q}_p), \mathbf{C}^\times)$, satisfying

$$e_p(\chi, t_p^\chi) = 1.$$

For almost all primes p the spherical vector

$$t_p^0 \in \mathcal{W}(\Pi_p, \psi_p)^{G(\mathbf{Z}_p)}$$

does the job. This is at least the case if Π_p and χ_p are unramified, ψ_p has conductor $\mathcal{O} \otimes \mathbf{Z}_p$, and

$$\int_{H(\mathbf{Z}_p)} dh_p = 1,$$

which is always satisfied for almost all primes p .

3.3 The global zeta integral

By the preceding discussion we may choose for each global quasi-character

$$\chi : H(\mathbf{Q}) \backslash H(\mathbf{A}) \rightarrow \mathbf{C}^\times,$$

a factorizable vector

$$t^\chi = \otimes_v t_v^{\chi_v} \in \mathcal{W}(\Pi, \psi)^{(K)},$$

where v runs through the finite places of \mathbf{Q} and for almost all (finite) places v ,

$$t_v^{\chi_v} = t_v^0.$$

Then the inverse Fourier transform

$$\Theta(t^\chi) : g \mapsto \sum_{\gamma \in G_n(\mathbf{Q}) \times G_{n-1}(\mathbf{Q})} t^\chi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot g \right) \in \Pi^{(K)}$$

gives rise to the global integral representation

$$\Lambda(s, \Pi \otimes \chi) = I(s, \chi, \Theta(t^\chi)) := \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \Theta(t^\chi)(h) \chi \otimes \omega_{s-\frac{1}{2}}^0(h) dh,$$

of the completed L -function which converges absolutely for every $s \in \mathbf{C}$ and thus defines an entire function in s . For s in a suitable right half plane the global integral decomposes into the infinite Euler product of the local zeta integrals (23) and (26), and we write $L(s, \Pi \otimes \chi)$ for the corresponding incomplete L -function.

4 Period relations

In this section we proof the expected period relations, that we will compare to Deligne's Conjecture in the motivic context in the last section.

4.1 Arithmeticity conditions

Following the terminology of [24, Section 3], adapted to the totally real case, we call an absolutely irreducible rational G -module M over E/\mathbf{Q} *arithmetic*, if it is essentially self-dual over \mathbf{Q} , i.e.

$$M^\vee \cong M \otimes \xi,$$

with a \mathbf{Q} -rational character $\xi \in X_{\mathbf{Q}}(G)$. We remark that since F is totally real, all its Galois twists M^τ , $\tau \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are arithmetic if M is so.

The center Z of G may be naturally identified with $G_1 \times G_1$. We may identify it with a factor of the maximal torus $T \subseteq G$, i.e.

$$T = T^{\text{der}} \cdot (G_1 \times G_1),$$

where

$$T^{\text{der}} = T \cap G^{\text{der}}.$$

The decomposition (5) of G_1 induces a natural projection

$$p_T : T \rightarrow T/(T^{\text{der}} \cdot (G_1^{\text{an}} \times G_1^{\text{an}})) =: Z^s \cong \text{GL}_1 \times \text{GL}_1,$$

and thus a monomorphism

$$p_T^* : X_{\mathbf{Q}}(Z^s) \rightarrow X_{\mathbf{Q}}(T).$$

We fix the identification

$$Z^s = \text{GL}_1 \times \text{GL}_1,$$

once and for all in such a way that each factor of GL_1 corresponds to the corresponding factor of the center of G . This gives us an identification

$$X_{\mathbf{Q}}(Z^s) = \mathbf{Z}^2,$$

and we simply write (w_1, w_2) for the image of $(w_1, w_2) \in X_{\mathbf{Q}}(Z^s)$ under p_T^* .

If M is of highest weight μ , we denote by μ^\vee the highest weight of M^\vee . Then M is arithmetic if and only if

$$\mu^\vee - \mu \in \text{im}(p_T^*).$$

Then

$$(\mu^\sigma)^\vee - \mu^\sigma = (w_1, w_2), \tag{28}$$

with (w_1, w_2) independent of $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. We may think of (w_1, w_2) as a (pair of) weight(s) in the motivic sense.

A \mathbf{Q} -rational character $\xi \in X_{\mathbf{Q}}(H) \cong \mathbf{Z}$ is called *critical* for M if

$$\text{Hom}_H(M, \xi) \neq 0.$$

Due to the multiplicity one property of the pair $\text{GL}_{n+1} | \text{GL}_n$ (cf. the non-compact analogue of Proposition 4.1 below) we have

$$\dim_{\mathbf{Q}(\mu)} \text{Hom}_H(M_{\mathbf{Q}(\mu)}, \xi_{\mathbf{Q}(\mu)}) = 1, \tag{29}$$

for each critical ξ . We call M *critical* if it admits a critical character ξ .

4.2 Rational test vectors

We specialize the notation of section 2 to the case

$$\begin{aligned} n_1 &= \begin{cases} n, & 2 \mid n, \\ n+1, & 2 \nmid n, \end{cases} \\ n_2 &= \frac{(n+1)n}{n_1}, \end{aligned}$$

and $r = 2$. Hence n_1 is even and n_2 is odd. We choose $E_K = \mathbf{Q}(\sqrt{-1})$ and fix a θ -stable Borel $\mathfrak{q} \subseteq \mathfrak{g}$, which is a product of two θ -stable parabolic subalgebras in each factor of $G = G_{n+1} \times G_n$. We assume \mathfrak{q} to be transversal to the Lie algebra \mathfrak{h} of H , i.e.

$$\mathfrak{g}_{E_K} = \mathfrak{q}_{E_K} \oplus \mathfrak{h}_{E_K}. \quad (30)$$

The existence of such a \mathfrak{q} was explained in section 1.2.

We assume that Π_∞ has non-trivial relative Lie algebra cohomology with coefficients in $M(\mu)^\vee$. We know that $M(\mu)$ is arithmetic if Π exists. We know by [49] that we have an isomorphism

$$\iota_\infty : A_{\mathfrak{q}}(\mu)_{\mathbf{C}} \rightarrow \mathscr{W}(\Pi_\infty, \Psi_\infty)^{(K)} \quad (31)$$

where the base change on the left hand side is implicitly understood via the fixed embedding (13).

For any archimedean place v of F we choose an $l_v \in K_n(\mathbf{Q})$ with the property that its image in

$$\pi_0(K_n) \cong \{\pm 1\}^{r_F}$$

is non-trivial in the factor $\{\pm 1\}$ corresponding to v and trivial in the factors corresponding to other real places. We assume without loss of generality that l_v is chosen in the image of a homomorphic section

$$\pi_0(K_n) \rightarrow K_n.$$

We consider l_v as an element of $L(\mathbf{Q})$ via the respective identification

$$G_n = L. \quad (32)$$

Then the image of l_v in $\pi_0(K)$ corresponds likewise under the isomorphism

$$\pi_0(K) \cong \{\pm 1\}^{r_F} \times \{\pm 1\}^{r_F},$$

to the element whose v -component is $(-1, -1)$ and trivial otherwise. Therefore the action of l_v on $B_{\mathfrak{q}}(\mu)$ interchanges the two modules $B_{\mu_{v,1}} \otimes B_{\mu_{v,2}}$ and $B_{\tilde{\mu}_{v,1}} \otimes B_{\mu_{v,2}}$ and leaves the other factors $B_{\mu_{v',1}} \otimes B_{\mu_{v',2}}$, $v' \neq v$, in (19) invariant. In particular we see that any irreducible K^0 -submodule of $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ generates $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ as a representation of L and even of $L \cap H_1$.

The transversality condition (30) gives us

$$\mathfrak{k}_{\mathbf{C}} = \mathfrak{l}_{\mathbf{C}} \oplus (\mathfrak{q}_{\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}})$$

and by the Poincaré-Birkhoff-Witt Theorem we have an isomorphism

$$U(\mathfrak{k}_{\mathbf{C}}) = U(\mathfrak{l}_{\mathbf{C}}) \otimes U(\mathfrak{q}_{\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}}) \quad (33)$$

of \mathbf{C} -vector spaces. As a consequence we obtain

Proposition 4.1. *For each character $\chi \in X_{\mathbf{C}}(L)$ we have*

$$\dim_{\mathbf{C}} \operatorname{Hom}_L(B_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}}) \leq 1.$$

Proof. We sketch a proof for the convenience of the reader, which is an adaption of an argument in [47, Lemma 2.10]. We show first the analogous statement

$$\dim_{\mathbf{C}} \operatorname{Hom}_{L^0}(B_{\mathfrak{q}}^{\circ}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}}) \leq 1.$$

To this point, let

$$\lambda \in \operatorname{Hom}_{L^0}(B_{\mathfrak{q}}^{\circ}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}}).$$

Assume that the restriction of λ to the one-dimensional highest weight space

$$H^0(\mathfrak{u}_{\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}}; B_{\mathfrak{q}}^{\circ}(\mu)_{\mathbf{C}})$$

vanishes. Since any non-zero vector of this space generates $B^{\circ}(\mu)_{\mathbf{C}}$ as a $U(\mathfrak{k}_{\mathbf{C}})$ -module, we see with (33) that λ must vanish. Hence the space of L^0 -equivariant $\lambda : B_{\mathfrak{q}}^{\circ}(\mu) \rightarrow \chi$ is at most one-dimensional.

Since $B_{\mathfrak{q}}^{\circ}(\mu)_{\mathbf{C}}$ generates $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ as an L -module, the claim follows. \square

Proposition 4.2. *For $n \geq 1$ there exists a vector*

$$t_0 \in B_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$$

with the property that for any character $\chi \in X_{\mathbf{C}}(L)$ and any

$$0 \neq \lambda \in \operatorname{Hom}_{L^0}(B_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}})$$

we have

$$\lambda(t_0) \neq 0.$$

Proof. We first remark that, since $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ is completely reducible as an L -module, there are only finitely many characters $\chi \in X_{\mathbf{C}}(L)$ with the property

$$\operatorname{Hom}_L(B_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}}) \neq 0. \quad (34)$$

By Proposition 4.1 the union of the kernels of all non-zero functionals λ for the finitely many $\chi \in X_{\mathbf{C}}(L)$ satisfying (34) is a Zariski closed subset of $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ of codimension 1. In particular its complement $U \subseteq B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ is non-empty and open for the standard topology on $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ as a finite-dimensional topological \mathbf{C} -vector space. Since $B_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$ is dense in $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$, we find an element

$$t_0 \in U \cap B_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)},$$

as desired. This concludes the proof. \square

In representation theoretic terms the main theorem of [30] is

Theorem 4.3 (Kasten-Schmidt [30, Theorem 4.4]). *Let $n = 2$ and F be a totally real number field. Then, for any quasi-character χ of $H(\mathbf{R})$ the restriction map*

$$\mathrm{Hom}_{\mathfrak{h},L}(A_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}}) \rightarrow \mathrm{Hom}_L(B_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi|_{L,\mathbf{C}}) \quad (35)$$

is a monomorphism.

Strictly speaking Kasten and Schmidt only discuss the case $F = \mathbf{Q}$. But their method is entirely local and therefore immediately generalizes to the totally real case. This follows for example from the structure of the bottom layer (18), together with the analogous statement for $A_{\mathfrak{q}}(\mu)_{\mathbf{C}}$.

As an immediate consequence of Theorem 4.3, we obtain

Proposition 4.4. *Let $1 \leq n \leq 2$ and let F a totally real number field. Then for any quasi-character χ of $H(\mathbf{R})$ we have*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{h},L}(A_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}}) = 1. \quad (36)$$

By Proposition 1.1 in [26] the two Hom spaces in (35) are defined over the field of definition of the modules under consideration. Writing $\mathbf{Q}(\mu, \chi)$ for the composite of $\mathbf{Q}(\mu)$ and the field of definition $\mathbf{Q}(\chi)$ of the quasi-character χ , as a fixed subfield of \mathbf{C} , we obtain

Proposition 4.5 (Proposition 1.1 in [26]). *For any quasi-character χ of $H(\mathbf{R})$ we have*

$$\mathrm{Hom}_{\mathfrak{h},L}(A_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi_{\mathbf{C}}) = \mathrm{Hom}_{\mathfrak{h},L}(A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu, \chi)}, \chi_{\mathbf{Q}(\mu, \chi)}) \otimes_{\mathbf{Q}(\mu, \chi)} \mathbf{C}.$$

Each one-dimensional $(\mathfrak{h}, L)_{\mathbf{C}}$ -module $\chi_{\mathbf{C}}$ corresponds bijectively to a quasi-character χ of $H(\mathbf{R})$ and by composing the functional $e_{\infty}(\chi, \cdot)$ of section 3.1 with the fixed isomorphism ι_{∞} in (31) we obtain a non-zero $(\mathfrak{h}, L)_{\mathbf{C}}$ -equivariant functional

$$\lambda_{\chi, \mathbf{C}} := e_{\infty}(\chi, \cdot) \circ \iota_{\infty} : A_{\mathfrak{q}}(\mu)_{\mathbf{C}} \rightarrow \chi_{\mathbf{C}}^{-1}. \quad (37)$$

We formulate the following

Conjecture 4.6. *For each $n \geq 3$ and each quasi-character χ the functional $\lambda_{\chi, \mathbf{C}}$ is non-zero on the minimal K -type $B_{\mathfrak{q}}(\mu)_{\mathbf{C}}$ and $\lambda_{\chi, \mathbf{C}}$ is defined over $\mathbf{Q}(\mu)$ (up to a complex unit) whenever χ is algebraic.*

For the algebraicity statement in Conjecture 4.6 we may restrict ourselves to critical χ , but by our current understanding of the situation this does not make a difference, since it is possible to write down a rational candidate for $\lambda_{\chi, \mathbf{C}}$, cf. [27].

We also formulate the slightly weaker

Conjecture 4.7. *For each $n \geq 3$, there is a good rational test vector $t \in A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$, which is good up to a complex unit, and for each critical quasi-character χ , $\lambda_{\chi, \mathbf{C}}$ is defined over $\mathbf{Q}(\mu)$.*

The argument in the proof of Proposition 4.2 together with Theorem 4.8 below shows that Conjecture 4.6 implies Conjecture 4.7.

We remark that by our preceding discussion the statement of Conjecture 4.6 is true for $1 \leq n \leq 2$, and the problem over a totally F/\mathbf{Q} reduces to the case $F = \mathbf{Q}$. The results of Sun [46] show that $\lambda_{\chi, \mathbf{C}}$ is always non-zero on the minimal K -type for critical χ , but this is still far from the same statement for *all* quasi-characters χ . Conjecture 4.6 would be implied by the generalization of Theorem 4.3 to arbitrary $n \geq 3$, but such a strong claim may not be true in general. The statement in Conjecture 4.7 is considerably weaker. Finally we remark that a proof of Conjecture 4.6 would automatically provide another proof of the Non-vanishing Hypothesis, whereas Conjecture 4.7 finds only applications here thanks to Sun's [46].

For $n \geq 3$ our results in the rest of the paper depend conditionally on Conjecture 4.7.

The unconditional case $n = 2$ corresponds to $G = \text{Res}_{F/\mathbf{Q}}(\text{GL}_3 \times \text{GL}_2)$, in which case our results are new. The case $n = 1$ corresponds to the case of Hilbert modular forms and gives a new proof of the well known period relations in that case, which are due to Manin and Shimura [35, 36, 42, 43, 44].

Theorem 4.8. *Assume $1 \leq n \leq 2$ or that Conjecture 4.6 is true. Then there exists a rational test vector $t_0 \in B_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$ with the property that for every $\chi \in \text{Hom}(H(\mathbf{R}), \mathbf{C}^\times)$,*

$$\lambda_{\chi, \mathbf{C}}(t_0) \neq 0. \quad (38)$$

In particular for every quasi-character χ there exists a constant $c_\chi \in \mathbf{C}^\times$, only depending on the connected component containing χ in $\text{Hom}(H(\mathbf{R}), \mathbf{C}^\times)$, with the property that

$$e_\infty(\chi, \iota_\infty(t_0)) = c_\chi.$$

Proof. By Theorem 4.3 we know

$$0 \neq \lambda_{\chi, \mathbf{C}}|_{B_{\mathfrak{q}}(\mu)_{\mathbf{C}}} \in \text{Hom}_L(B_{\mathfrak{q}}(\mu)_{\mathbf{C}}, \chi|_{L, \mathbf{C}}^{-1}). \quad (39)$$

Therefore any choice of t_0 as in Proposition 4.2 satisfies (38). By the archimedean Rankin-Selberg theory that we discussed in section 3.1, the condition

$$\forall \chi: \quad e_\infty(\chi, \iota_\infty(t_0)) \neq 0$$

is satisfied for all χ and this implies the claim. \square

Corollary 4.9. *For any $t \in A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$ and any algebraic quasi-character $\chi \in X^{\text{alg}}(H(\mathbf{R}))$ we have*

$$e_\infty(\chi, \iota_\infty(t)) \in \mathbf{Q}(\mu) \cdot c_\chi.$$

Proof. Since χ is defined over \mathbf{Q} , we see with Propositions 4.4 and 4.5 that the functional (37) is defined over $\mathbf{Q}(\mu)$. This in turn implies that the image of the rational subspace

$$A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)} \subseteq A_{\mathfrak{q}}(\mu)_{\mathbf{C}}$$

under $\lambda_{\chi, \mathbf{C}}$ is a one-dimensional $\mathbf{Q}(\mu)$ -subspace of $\chi_{\mathbf{C}}^{-1} \cong \mathbf{C}$. This subspace contains $\lambda_{\chi, \mathbf{C}}(t)$ and $\lambda_{\chi, \mathbf{C}}(t_0)$. Therefore the claim follows from Theorem 4.8. \square

4.3 The archimedean period relation

Our first result towards period relations is

Theorem 4.10. *For any algebraic quasi-character χ of $H(\mathbf{R})$ we have*

$$c_{\text{sgn}_\infty \otimes \chi} \in \mathbf{Q}(\mu) \cdot (i^{r_F m} \cdot c_\chi),$$

where

$$m := \frac{(n+1)n}{2}.$$

Proof. According to the direct sum decomposition (3), or equivalently (18), which is already defined over $E_K(\mu)$, we may write

$$t := t_0 = \sum_{\varepsilon \in \pi_0(H)} t_\varepsilon \tag{40}$$

where

$$t_\varepsilon \in \varepsilon \cdot B_{\mathbf{q}}^\circ(\mu)_{E_K(\mu)} \subseteq B_{\mathbf{q}}(\mu)_{E_K(\mu)},$$

for our choice of representatives

$$\varepsilon = \prod_v l_v^{\delta_v} \in L(\mathbf{Q}), \quad \delta_v \in \{0, 1\}.$$

Since these elements form a system of representatives of

$$\pi_0(H(\mathbf{R})) = \pi_0(G(\mathbf{R})/C(\mathbf{R})),$$

we have

$$G(\mathbf{R}) = \bigsqcup_{\varepsilon} G(\mathbf{R})^0 C(\mathbf{R}) \cdot \varepsilon.$$

Considering $A_{\mathbf{q}}(\mu)_{\mathbf{C}}$ as a $(\mathfrak{g}, K^0)_{\mathbf{C}}$ -module, we see that $\iota_\infty(t_\varepsilon)$, as a function on $G(\mathbf{R})$, has support in a set of the form $G(\mathbf{R})^0 C(\mathbf{R}) \cdot \varepsilon'$, and

$$\varepsilon' = \varepsilon \cdot \varepsilon_0,$$

with a representative ε_0 independent of ε . We remark that $\varepsilon_0 = 1$ thanks to (15), which is compatible with parabolic induction.

Accordingly the archimedean Rankin-Selberg integral (23) decomposes into the sum of the integrals

$$\Psi_\infty^\varepsilon : (\chi, w) \mapsto \int_{N(\mathbf{R}) \backslash H(\mathbf{R})^0 \varepsilon} w(h_\infty) \chi(h_\infty) dh_\infty,$$

over the individual connected components.

These integrals are convergent for χ in a suitable right half plane. Similarly we have the ratios

$$e_\infty^\varepsilon : (\chi, w) \mapsto \frac{\Psi_\infty^\varepsilon(\chi, w)}{L(\frac{1}{2}, \Pi_\infty \otimes \chi)},$$

which are again entire functions in χ , and

$$e_\infty(\chi, \iota_\infty(t)) = \sum_\varepsilon e_\infty^\varepsilon(\chi, \iota_\infty(t_\varepsilon)).$$

Since dh_∞ is a Haar measure, we get the relation

$$e_\infty^\varepsilon(\chi, \iota_\infty(t)) = \chi(\varepsilon) \cdot e_\infty^1(\chi|_{H(\mathbf{R})^0}, \varepsilon \cdot \iota_\infty(t_\varepsilon)). \quad (41)$$

With relation (41) we conclude that for each real place v of F ,

$$\begin{aligned} \lambda_{\chi, E_K(\mu)}(t_\varepsilon) &= e_\infty^\varepsilon(\chi, \iota_\infty(t)) \\ &= \chi(\varepsilon) \cdot e_\infty^1(\chi|_{H(\mathbf{R})^0}, \varepsilon \cdot \iota_\infty(t_\varepsilon)) \\ &= \chi(\varepsilon) \cdot e_\infty^1((\chi \otimes \text{sgn}_v)|_{H(\mathbf{R})^0}, \varepsilon \cdot \iota_\infty(t_\varepsilon)) \\ &= \text{sgn}_v(\varepsilon) \cdot e_\infty^\varepsilon(\chi \otimes \text{sgn}_v, \iota_\infty(t_\varepsilon)) \\ &= \text{sgn}_v(\varepsilon) \cdot \lambda_{\chi \otimes \text{sgn}_v, E_K(\mu)}(t_\varepsilon). \end{aligned}$$

Summing up, we obtain

$$\lambda_{\chi, E_K(\mu)}(t) = \sum_\varepsilon \lambda_{\chi, E_K(\mu)}(t_\varepsilon) = \sum_\varepsilon \text{sgn}_v(\varepsilon) \cdot \lambda_{\chi \otimes \text{sgn}_v, E_K(\mu)}(t_\varepsilon). \quad (42)$$

Complex conjugation $\tau_K \in \text{Gal}(E_K/\mathbf{Q})$ leaves the direct sum decompositions (3) and (18) invariant, but permutes the direct factors. By Theorem 2.3 this action is trivial if and only if $2 \mid m$, i.e. if and only if

$$i^m \in \mathbf{Q}.$$

Let us suppose $\sqrt{-1} \notin \mathbf{Q}(\mu)$ and $2 \nmid m$. Choose a real place v_0 of F and consider the K^0 -submodule

$$B_{v_0, E_K(\mu)} := \sum_{\varepsilon \in \ker \text{sgn}_{v_0}} \varepsilon \cdot B_{\mathbf{q}}^\circ(\mu)_{E_K(\mu)} \subseteq B_{\mathbf{q}}(\mu)_{E_K(\mu)},$$

where the sum ranges over all possible products ε of the elements l_v with $v \neq v_0$. Then

$$B_{\mathbf{q}}(\mu)_{E_K(\mu)} = B_{v_0, E_K(\mu)} \oplus l_{v_0} \cdot B_{v_0, E_K(\mu)}. \quad (43)$$

The second direct summand on the right hand side is naturally identified with the dual of $B_{v_0, E_K(\mu)}$ due to our hypothesis $2 \nmid m$. We conclude that τ_K , as an automorphism of $E_K(\mu)/\mathbf{Q}(\mu)$, and thus of $B_{\mathbf{q}}(\mu)_{E_K(\mu)}$, interchanges the two direct summands in (43).

Hence τ_K sends the vector

$$t_{v_0} := \sum_{\varepsilon \in \ker \operatorname{sgn}_{v_0}} t_\varepsilon$$

to

$$t_{v_0}^{\tau_K} \in l_{v_0} \cdot B_{v_0, E_K(\mu)}.$$

Now t is $\mathbf{Q}(\mu)$ -rational, and thus invariant under τ_K . The sum decomposition (40) being unique, we conclude that for each representative ε ,

$$t_{v_0}^{\tau_K} = \sum_{\varepsilon \in \ker \operatorname{sgn}_{v_0}} t_{l_{v_0}\varepsilon} =: t_{-v_0}.$$

Hence the vector

$$t_{v_0} - t_{-v_0} \in i \cdot B_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)} \subseteq B_{\mathfrak{q}}(\mu)_{E_K(\mu)}.$$

is ‘purely imaginary’ (in the Galois theoretic sense in the context of the non-trivial extension $E_K(\mu)/\mathbf{Q}(\mu)$), and consequently

$$i \cdot (t_{v_0} - t_{-v_0}) \in B_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)} \subseteq B_{\mathfrak{q}}(\mu)_{E_K(\mu)}. \quad (44)$$

Turning our attention to the functionals in (42), we observe

$$\begin{aligned} \lambda_{\chi, E_K(\mu)}(t) &= \lambda_{\chi \otimes \operatorname{sgn}_v, E_K(\mu)}(t_{v_0}) - \lambda_{\chi \otimes \operatorname{sgn}_v, E_K(\mu)}(t_{-v_0}) \\ &= \lambda_{\chi \otimes \operatorname{sgn}_v, E_K(\mu)}(t_{v_0} - t_{-v_0}) \\ &= i \cdot \lambda_{\chi \otimes \operatorname{sgn}_v, E_K(\mu)}(i \cdot (t_{-v_0} - t_{v_0})) \\ &\in \lambda_{\chi \otimes \operatorname{sgn}_v, \mathbf{Q}(\mu)}(t) \cdot i \cdot \mathbf{Q}(\mu), \end{aligned}$$

where the last relation follows from 44 and the rationality property of the functional. Since

$$\operatorname{sgn}_\infty = \otimes_{v_0} \operatorname{sgn}_{v_0},$$

iteration over the r_F real places of F proves the claim in the case $\sqrt{-1} \notin \mathbf{Q}(\mu)$ and $2 \nmid m$.

If $\sqrt{-1} \in \mathbf{Q}(\mu)$ or $2 \mid m$, then the vectors t and $t_{\pm v_0}$ all lie in the same $\mathbf{Q}(\mu)$ -rational model $B(\mu)_{\mathbf{Q}(\mu)}$, and thus the claim follows in this case by the rationality of the functional as well. \square

The proof of Theorem 4.10 may be interpreted as an automorphic reflection of the motivic Corollaire 1.6 in [11]. We will discuss this relation in more detail in section 5.

Corollary 4.11. *For any $t \in A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$, and any algebraic quasi-character χ of $H(\mathbf{R})$ and any $k \in \mathbf{Z}$ we have*

$$\lambda_{\chi[k], \mathbf{Q}(\mu)}(t) \in \mathbf{Q}(\mu) \cdot (i^{kr_F m} \cdot c_\chi).$$

Proof. By (6) the two quasi-characters

$$\chi[k] = \chi \otimes (\mathcal{N}^{\otimes k})$$

and χ lie in the same connected component if and only if $2 \mid k$. In the case $2 \nmid k$ the character $\chi[k]$ lies in the same component as $\chi \otimes \text{sgn}_\infty$. The corollary follows from Theorem 4.10 and the constancy of c_χ and $c_{\chi \otimes \text{sgn}_\infty}$ on connected components (Theorem 4.8). \square

We say that an algebraic $\chi = \text{sgn}^\delta \otimes (\mathcal{N}^{\otimes k})$ is *critical* for Π (or Π_∞), if $L(s, \Pi_\infty)$ and $L(1-s, \Pi_\infty^\vee)$ both have no pole at $s = k + \frac{1}{2}$. For critical χ we know that

$$L_\infty\left(\frac{1}{2}, \Pi_\infty \otimes \chi\right) \neq 0.$$

By Theorem 4.8 we therefore find a $t_0 \in A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$ satisfying

$$\Psi_\infty(\chi, \iota_\infty(t_0)) \neq 0,$$

for any critical χ , $\Psi_\infty(\chi, \iota_\infty(t_0))$ being defined by holomorphic continuation outside the region of absolute convergence. This implies

Corollary 4.12. *For any $t \in A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$, any pair of critical quasi-characters χ, χ' of $H(\mathbf{R})$ with*

$$\chi' = \chi[k] \otimes (\text{sgn}_\infty)^\delta, \quad k \in \mathbf{Z}, \delta \in \{0, 1\},$$

we have

$$\Psi_\infty(\chi', \iota_\infty(t)) \in \mathbf{Q}(\mu) \cdot i^{(k+\delta)r_{F^m}} \cdot \Psi_\infty(\chi, \iota_\infty(t_0)) \cdot \frac{L_\infty(\frac{1}{2} + k, \Pi_\infty \otimes \chi)}{L_\infty(\frac{1}{2}, \Pi_\infty \otimes \chi)},$$

for every $t \in A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}$.

4.4 Cohomology and Galois actions

For any $\sigma \in \text{Aut}(\mathbf{C}/\mathbf{Q})$ we have a twisted representation

$$M(\mu)_{E^\sigma}^\sigma := M(\mu)_E \otimes_{E, \sigma^{-1}} \sigma^{-1}(E)$$

of G . This twisting operation is compatible with the Galois action (9) on highest weights. We have the counit map

$$\epsilon : M(\mu) \rightarrow (\text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}} M(\mu)) \otimes_{\mathbf{Q}} \mathbf{Q}(\mu),$$

where the right hand side is defined over \mathbf{Q} . This map extends to an E -linear map

$$\epsilon_E : M(\mu)_E \rightarrow (\text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}} M(\mu)) \otimes_{\mathbf{Q}} E,$$

which fits into a commutative diagram

$$\begin{array}{ccc}
M(\mu)_E & \xrightarrow{\epsilon_E} & (\text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}} M(\mu))_E \\
\sigma^{-1} \downarrow & & \downarrow \sigma^{-1} \\
M(\mu)_{E^\sigma}^\sigma & \xrightarrow{\epsilon_E} & (\text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}} M(\mu))_{E^\sigma}
\end{array}$$

In other words the Galois action on $M(\mu)$ is compatible with the intrinsic Galois action on the \mathbf{C} -valued points of the \mathbf{Q} -rational module $\text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}} M(\mu)$. In particular the latter may be thought of as the sum of the Galois conjugates of $M(\mu)$.

Now the same discussion applies mutatis mutandis to $A_{\mathfrak{q}}(\mu)$ and $A_{\mathfrak{q}}(\mu) \otimes M(\mu)^\vee$ instead of $M(\mu)$. Note that restriction of scalars does not commute with tensor products, but we have natural isomorphisms of (\mathfrak{g}, K) -modules over $\mathbf{Q}(\mu)$:

$$\begin{aligned}
\text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}}(A_{\mathfrak{q}}(\mu) \otimes M(\mu)^\vee)_{\mathbf{Q}(\mu)} &= \text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}}(A_{\mathfrak{q}}(\mu))_{\mathbf{Q}(\mu)} \otimes M(\mu)^\vee \\
&= A_{\mathfrak{q}}(\mu) \otimes \text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}}(M(\mu)^\vee)_{\mathbf{Q}(\mu)}.
\end{aligned}$$

We introduce the \mathbf{Q} -group

$$GK = \{g \in G \mid \exists z \in Z^s : zg = \theta(g)\} \subseteq G.$$

It is the product of K with the maximal \mathbf{Q} -split torus in the center Z of G . The natural isomorphism

$$K/K^0C = GK/GK^0C = \pi_0(L),$$

in light of (15), Shapiro's Lemma implies

$$\begin{aligned}
H^\bullet(\mathfrak{g}, GK^0; A_{\mathfrak{q}}(\mu) \otimes M(\mu)^\vee)_{E_K(\mu)} &= H^\bullet(\mathfrak{g}, GK^0; \Gamma_{\mathfrak{g}, CK^0}^{\mathfrak{g}, K}(A_{\mathfrak{q}}^\circ(\mu) \otimes M(\mu)^\vee))_{E_K(\mu)} \\
&= \text{Ind}_{\pi_0(L^0)}^{\pi_0(L)} H^\bullet(\mathfrak{g}, GK^0; A_{\mathfrak{q}}^\circ(\mu) \otimes M(\mu)^\vee)_{E_K(\mu)} \\
&= H^\bullet(\mathfrak{g}, GK^0; A_{\mathfrak{q}}^\circ(\mu) \otimes M(\mu)^\vee)_{E_K(\mu)} \otimes \mathbf{Q}[\pi_0(L)] \\
&=: H^\bullet(\mathfrak{g}, GK^0; A_{\mathfrak{q}}^\circ(\mu) \otimes M(\mu)^\vee)[\pi_0(L)]_{E_K(\mu)},
\end{aligned}$$

as $\pi_0(L)$ -modules.

Introduce the bottom degree

$$b_n^{\mathbf{R}} := \left\lfloor \frac{n^2}{4} \right\rfloor,$$

which is the lowest degree for which the relative Lie algebra cohomology of non-degenerate cohomological representations of $\text{GL}_n(\mathbf{R})$ does not vanish, and set

$$d := \sum_v b_{n+1}^{F_v} + b_n^{F_v}.$$

This is the bottom degree of Lie algebra cohomology for non-degenerate cohomological representations of $G(\mathbf{R})$. Since the cohomology of $A_{\mathfrak{q}}^{\circ}(\mu)$ in the degree d is one-dimensional, the standard descent argument [26, Proposition 1.1] together with the Homological Base Change Theorem in loc. cit. shows that we have a natural isomorphism of $\pi_0(L)$ -modules

$$H^d(\mathfrak{g}, GK^0; A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)} = \mathbf{Q}(\mu)[\pi_0(L)]. \quad (45)$$

Applying the same restriction of scalars argument again, we see that

$$H^d(\mathfrak{g}, GK^0; \text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}}(A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})) = \text{Res}_{\mathbf{Q}(\mu)/\mathbf{Q}} \mathbf{Q}(\mu)[\pi_0(L)]$$

is defined over \mathbf{Q} . Over \mathbf{C} it comes with a natural action of $\text{Aut}(\mathbf{C}/\mathbf{Q})$, and this action has an extension to a global \mathbf{Q} -structure on the space of regular algebraic cusp forms, cf. Theorem A of [26].

4.5 Cohomological test vectors

We have a natural isomorphism

$$H^{\bullet}(\mathfrak{g}, GK^0; A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)} = H^0(GK^0; \bigwedge^{\bullet}(\mathfrak{g}/\mathfrak{g}\mathfrak{k})^{\vee} \otimes A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)} \quad (46)$$

of $\pi_0(L)$ -modules, since the standard complex computing the relative Lie algebra cohomology degenerates in our case. This is well known over \mathbf{C} (cf. combine Proposition 3.1 in Borel-Wallach [7] with Proposition 3.1 in [26]), and this already implies the claim over $\mathbf{Q}(\mu)$.

The structure of (\mathfrak{g}, K) -cohomology has been studied in general by Vogan and Zuckerman in [49]. In particular we know that the canonical embedding

$$H^0(GK^0; \bigwedge^{\bullet}(\mathfrak{g}/\mathfrak{g}\mathfrak{k})^{\vee} \otimes B_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)} \rightarrow H^0(GK^0; \bigwedge^{\bullet}(\mathfrak{g}/\mathfrak{g}\mathfrak{k})^{\vee} \otimes A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)}$$

is an isomorphism. But we won't need this.

Any cohomology class

$$h \in H^d(\mathfrak{g}, GK^0; A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)}$$

has by (46) a unique representative

$$h = \sum_{p=1}^s \omega_p \otimes a_p \otimes m_p \in H^0(GK^0; \bigwedge^d(\mathfrak{g}/\mathfrak{g}\mathfrak{k})^{\vee} \otimes A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)},$$

with

$$\omega_p \in \bigwedge^d(\mathfrak{g}/\mathfrak{g}\mathfrak{k})_{\mathbf{Q}(\mu)}^{\vee}, \quad a_p \in A_{\mathfrak{q}}(\mu)_{\mathbf{Q}(\mu)}, \quad m_p \in M(\mu)_{\mathbf{Q}(\mu)}^{\vee}, \quad 1 \leq p \leq s.$$

Identifying Z^s with the maximal \mathbf{Q} -split torus in the center of G and observe that

$$H \cap Z^s = 1.$$

We consider the diagonal embedding

$$(\mathfrak{h}/\mathfrak{l})_{\mathbf{Q}(\mu)} \rightarrow (\mathfrak{g}/\mathfrak{g}\mathfrak{k})_{\mathbf{Q}(\mu)},$$

which dually induces a projection

$$(\mathfrak{g}/\mathfrak{g}\mathfrak{k})_{\mathbf{Q}(\mu)}^\vee \rightarrow (\mathfrak{h}/\mathfrak{l})_{\mathbf{Q}(\mu)}^\vee.$$

The d -th exterior power of this map gives rise to the restriction map

$$\mathrm{Res}_H^G : \bigwedge^d (\mathfrak{g}/\mathfrak{g}\mathfrak{k})_{\mathbf{Q}(\mu)}^\vee \rightarrow \bigwedge^d (\mathfrak{h}/\mathfrak{l})_{\mathbf{Q}(\mu)}^\vee,$$

where the right hand side is one-dimensional due to the numerical coincidence

$$d = \dim \mathfrak{h}/\mathfrak{l}.$$

We fix a $\mathbf{Q}(\mu)$ -rational basis vector

$$0 \neq w_0 \in \bigwedge^d (\mathfrak{h}/\mathfrak{l})_{\mathbf{Q}(\mu)}.$$

The choice of w_0 amounts to choosing an isomorphism of vector spaces

$$\bigwedge^d (\mathfrak{h}/\mathfrak{l})_{\mathbf{Q}(\mu)}^\vee \rightarrow \mathbf{Q}(\mu), \quad \omega \mapsto \omega(w_0).$$

The left hand side is a one-dimensional L -module via the adjoint action on $\mathfrak{h}/\mathfrak{l}$, and we furnish the right hand side with an action of L such that the above map becomes L -linear. The resulting one-dimensional L -module is denoted \mathcal{L} . It is of finite order, more concretely $\mathcal{L}^{\otimes 2} \cong \mathbf{1}$.

Let us assume that the character $\mathcal{N}^{\otimes k}$ is critical for $M(\mu)^\vee$, and fix a non-zero element

$$0 \neq \xi_k \in \mathrm{Hom}_H(M(\mu)^\vee, \mathcal{N}^{\otimes k})_{\mathbf{Q}(\mu)}.$$

By [30, Theorem 2.3], we know that then all finite order twists

$$\chi = \mathcal{N}^{\otimes k} \otimes \mathrm{sgn}_\infty^\delta, \quad \delta \in \{0, 1\}^{r_F}$$

are critical quasi-characters of $H(\mathbf{R})$. Now for each such χ and any

$$\lambda \in \mathrm{Hom}_{\mathfrak{h}, C}(A_{\mathfrak{q}}(\mu), \chi^{-1})_{\mathbf{Q}(\mu)},$$

the $\mathbf{Q}(\mu)$ -rational functionals λ and ξ_k induce a $\mathbf{Q}(\mu)$ -rational $\pi_0(L)$ -equivariant map

$$H(\lambda \otimes \xi_k) : H^d(\mathfrak{g}, GK^0; A_{\mathfrak{q}}(\mu) \otimes M(\mu)^\vee)_{\mathbf{Q}(\mu)} \xrightarrow{\lambda \otimes \xi_k} H^d(\mathfrak{h}, ML^0; \chi^{-1}[k])_{\mathbf{Q}(\mu)}.$$

By Poincaré duality, our choice of vector w_0 induces an isomorphism

$$H^d(\mathfrak{h}, ML^0; \chi^{-1}[k])_{\mathbf{Q}(\mu)} \rightarrow (\mathcal{L} \otimes \text{sgn}_{\infty}^{\delta})_{\mathbf{Q}(\mu)},$$

of $\pi_0(L)$ -modules. The composition of the latter with $H(\lambda \otimes \xi_k)$, provides us with a $\pi_0(L)$ -equivariant map

$$I(\lambda \otimes \xi_k) : H^d(\mathfrak{g}, GK^0; A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)} \rightarrow (\mathcal{L} \otimes \text{sgn}_{\infty}^{\delta})_{\mathbf{Q}(\mu)},$$

which on the level of complexes is given explicitly by

$$h = \sum_{p=1}^s \omega_p \otimes a_p \otimes m_p \mapsto \sum_{p=1}^s \omega_p(w_0) \otimes \lambda(a_p) \otimes \xi_k(m_p).$$

4.6 The global period relation

Let $\chi = \otimes_v \chi_v$ be an algebraic Hecke character of F with

$$\chi_{\infty} = \mathcal{N}^{\otimes k(\chi)} \otimes \text{sgn}_{\infty}^{\delta(\chi)}$$

critical. The period $\Omega(\chi_{\infty}) \in \mathbf{C}^{\times}$ in the global special value formula (cf. Theorem 1.1 [37]),

$$\frac{\Lambda(\frac{1}{2}, \Pi \otimes \chi)}{G(\chi)^m \cdot \Omega(\chi_{\infty})} \in \mathbf{Q}(\mu, \chi)$$

arises as follows. We fix for each signature $\delta \in \{0, 1\}^{r_F}$ a generator

$$h_{\delta} \in H^d(\mathfrak{g}, GK^0; A_{\mathfrak{q}}(\mu) \otimes M(\mu)^{\vee})_{\mathbf{Q}(\mu)} = \mathbf{Q}(\mu)[\pi_0(L)]$$

of the generalized $\text{sgn}_{\infty}^{\delta}$ -eigenspace for the action of $\pi_0(L) = \pi_0(K_n)$. In our application to χ we'll specialize later to the δ satisfying the compatibility condition

$$\text{sgn}_{\infty}^{\delta} = \mathcal{L} \otimes \text{sgn}_{\infty}^{\delta(\chi) + k(\chi)}. \quad (47)$$

Now together with a choice of factorizable Whittaker vector

$$t^{(\infty)} = \otimes_v t_v \in \mathcal{W}(\Pi^{(\infty)}, \psi^{(\infty)}),$$

at the finite places $v \nmid \infty$, the cohomological vector h_{δ} gives rise to a cohomology class

$$t_{\delta} := \sum_{p=1}^s \omega_p \otimes (\iota_{\infty}(a_p) \otimes t^{(\infty)}) \otimes m_p \in H^d(\mathfrak{g}, GK^0; \mathcal{W}(\Pi, \psi)^{(K)} \otimes M(\mu)^{\vee})_{\mathbf{C}}.$$

Inverse Fourier transform turns this class into an automorphic cohomology class

$$\vartheta_{\delta} := \sum_{p=1}^s \omega_p \otimes \Theta(\iota_{\infty}(a_p) \otimes t^{(\infty)}) \otimes m_p \in H^d(\mathfrak{g}, GK^0; \Pi^{(K)} \otimes M(\mu)^{\vee})_{\mathbf{C}}.$$

Via the realization

$$\Pi \subseteq L_0^2(G(\mathbf{Q})Z(\mathbf{A})\backslash G(\mathbf{A}); \omega_\Pi),$$

this class gives then rise to a global cohomology class

$$c(t_\delta) \in H_c^d(G(\mathbf{Q})\backslash G(\mathbf{A})/GK(\mathbf{R})^0 K^{(\infty)}; \underline{M(\mu)^\vee}_{\mathbf{C}}),$$

with coefficients in the local system associated to $M(\mu)^\vee$, and a suitable compact open $K^{(\infty)}$ which is small enough that t' is $K^{(\infty)}$ -invariant and additionally the underlying orbifold is a manifold.

Now the cohomology with compact support carries a natural $\mathbf{Q}(\mu)$ -structure, inherited from the $\mathbf{Q}(\mu)$ -structure of the rational representation $M(\mu)$. By the work of Clozel [9], we know that $\Pi^{(\infty)}$ is defined over its field of rationality $\mathbf{Q}(\Pi)$. Since the map $t \mapsto c(t)$ is Hecke-equivariant, and since $\Pi^{(\infty)}$ occurs in degree d with multiplicity one by Matsushima's Formula, we may, under the assumption that t_δ is chosen in the natural $\mathbf{Q}(\Pi)$ -rational structure of $\mathscr{W}(\Pi^{(\infty)}, \Psi^{(\infty)})$ (cf. [14, 34, 38]), renormalize $c(t')$ via a scalar $\Omega(t_\delta) \in \mathbf{C}^\times$, such that

$$\Omega(t_\delta) \cdot c(t_\delta) \in H_c^d(G(\mathbf{Q})\backslash G(\mathbf{A})/GK(\mathbf{R})^0 K^{(\infty)}; \underline{M(\mu)}_{\mathbf{Q}(\Pi)}). \quad (48)$$

To each algebraic Hecke character χ over F , that we interpret as a character of $H(\mathbf{A})$ via composition with the determinant, we may associate a cohomology class as follows. We denote the corresponding character of H as $(k(\chi))$. Then we may attach to χ a rational cohomology class

$$c_\chi \in H^0(H(\mathbf{Q})\backslash H(\mathbf{A})/L(\mathbf{R})^0 L^{(\infty)}(\chi); \underline{(k(\chi))}_{\mathbf{Q}(\chi)}),$$

where $L^{(\infty)}(\chi) \subseteq K^{(\infty)} \cap H(\mathbf{A}^{(\infty)})$ is a compact open such that the finite order character χ factors over $\det(L^{(\infty)}(\chi))$.

Now the natural map

$$\begin{aligned} & H_c^d(G(\mathbf{Q})\backslash G(\mathbf{A})/GK(\mathbf{R})^0 K^{(\infty)}; \underline{M(\mu)^\vee}_{\mathbf{Q}(\Pi)}) \rightarrow \\ & H_c^d(H(\mathbf{Q})\backslash H(\mathbf{A})/L(\mathbf{R})^0 L^{(\infty)}; \underline{M(\mu)^\vee}_{\mathbf{Q}(\Pi)}) \rightarrow \\ & \xrightarrow{(-) \cup c_\chi} H_c^d(H(\mathbf{Q})\backslash H(\mathbf{A})/L(\mathbf{R})^0 L^{(\infty)}; \underline{M(\mu)^\vee \otimes (k(\chi))}_{\mathbf{Q}(\Pi, \chi)}), \end{aligned}$$

together with Poincaré duality for the right hand side, induces the modular symbol

$$H_c^d(G(\mathbf{Q})\backslash G(\mathbf{A})/GK(\mathbf{R})^0 K^{(\infty)}; \underline{M(\mu)^\vee}_{\mathbf{Q}(\Pi)}) \rightarrow H^0(\Gamma; M(\mu)^\vee \otimes (k(\chi))_{\mathbf{Q}(\Pi)}),$$

where $\Gamma \subseteq H(\mathbf{Q})$ is the arithmetic subgroup corresponding to $L^{(\infty)}$.

Composition of the modular symbol with ξ_k provides us with a $\pi_0(L)$ -equivariant map

$$H_c^d(G(\mathbf{Q})\backslash G(\mathbf{A})/GK(\mathbf{R})^0 K^{(\infty)}; \underline{M(\mu)}_{\mathbf{Q}(\Pi)}) \rightarrow (\mathcal{N}^{\otimes k+k(\chi)})_{\mathbf{Q}(\Pi)}.$$

The image of the normalized class (48) under this map is essentially the algebraic part of the critical L -value $L(\frac{1}{2} + k, \Pi \otimes \chi)$. To be more precise, the image of $\Omega(t_\delta)^{-1} \cdot c(t_\delta)$ under this map computes the global integral

$$\Omega(t_\delta)^{-1} \cdot \sum_{p=1}^r \omega_p(w_0) \cdot I(\frac{1}{2} + k, \chi, \Theta(\iota_\infty(a_p) \otimes t^{(\infty)})) \cdot \xi_k(m_p) \in \mathcal{N}_{\mathbf{C}}^{\otimes k},$$

an expression that vanishes whenever the compatibility condition (47) is violated, i.e. we may assume that

$$\text{sgn}_\infty^{\delta+k} = \mathcal{L} \otimes \text{sgn}_\infty^{\delta(\chi)+k(\chi)}.$$

Now by the non-archimedean period relation calculated by Raghuram-Shahidi in [38, Theorem 4.1], we know that we may find a $\mathbf{Q}(\pi, \chi)$ -rational $t^{(\infty)} \in \mathcal{W}(\Pi^{(\infty)}, \psi^{(\infty)})$, such that

$$e^{(\infty)}(\chi^{(\infty)}) \cdot | \cdot |_s^{\mathbf{A}^{(\infty)}}, t^{(\infty)}) = G(\chi)^m L(s, \Pi^{(\infty)} \otimes \chi^{(\infty)}),$$

where m is as before. In other words the image of the corresponding cohomology class $c(t_\delta)$ under the modular symbol and ξ_k computes the value

$$G(\chi)^m \cdot \Lambda(\frac{1}{2} + k, \Pi \otimes \chi) \cdot \sum_{p=1}^r \omega_p(w_0) \cdot e_\infty(\frac{1}{2} + k, \chi_\infty, \iota_\infty(a_p)) \cdot \xi_k(m_p).$$

Now the values a_p lie in the same $\mathbf{Q}(\mu)$ -rational subspace as the good test vector t , and by Corollary 4.12 we conclude the proof of the desired period relation. We obtain the global

Theorem 4.13. *Assume $1 \leq n \leq 2$ or that Conjecture 4.7 holds for n . Let F be a totally real number field, and (π, σ) be a pair of regular algebraic irreducible cuspidal automorphic representations of $\text{GL}_{n+1}(\mathbf{A}_F)$ and $\text{GL}_n(\mathbf{A}_F)$ respectively. Assume that $\pi_\infty \widehat{\otimes} \sigma_\infty$ has non-trivial Lie algebra cohomology with coefficients in an irreducible rational $G_{n+1} \times G_n$ -module M , which we assume to be critical. Denote by $s_0 = \frac{1}{2} + j_0$ the left most critical value of the Rankin-Selberg L -function $L(s, \pi \times \sigma)$ (such a s_0 exists). Then there exist non-zero periods Ω_\pm , numbered by the characters \pm of $\pi_0((F \otimes_{\mathbf{Q}} \mathbf{R})^\times)$, such that for each critical half integer $s_1 = \frac{1}{2} + j_1$ for $L(s, \pi \times \sigma)$, and each finite order Hecke character*

$$\chi : F^\times \backslash \mathbf{A}_F^\times \rightarrow \mathbf{C}$$

we have

$$\frac{L(s_1, (\pi \times \sigma) \otimes \chi)}{G(\overline{\chi})^{\frac{(n+1)n}{2}} \Omega_{(-1)^{j_1} \text{sgn } \chi}} \in \frac{L(s_0, \pi_\infty \times \sigma_\infty)}{L(s_1, \pi_\infty \times \sigma_\infty)} \cdot i^{j_1 r_F \frac{(n+1)n}{2}} \mathbf{Q}(\pi, \sigma, \chi).$$

Furthermore, for every $\tau \in \text{Aut}(\mathbf{C}/\mathbf{Q})$,

$$i^{-j_1 r_F \frac{(n+1)n}{2}} \frac{L(s_1, \pi_\infty \times \sigma_\infty)}{L(s_0, \pi_\infty \times \sigma_\infty)} \cdot \frac{L(s_1, (\pi^\tau \times \sigma^\tau) \otimes \chi^\tau)}{G(\overline{\chi^\tau})^{\frac{(n+1)n}{2}} \Omega_{(-1)^{j_1} \text{sgn } \chi^\tau}} =$$

$$\left(i^{-j_1 r_F \frac{(n+1)n}{2}} \frac{L(s_1, \pi_\infty \times \sigma_\infty)}{L(s_1, \pi_\infty \times \sigma_\infty)} \cdot \frac{L(s_0, (\pi \times \sigma) \otimes \chi)}{G(\overline{\chi})^{\frac{(n+1)n}{2}} \Omega_{(-1)^{j_1} \operatorname{sgn} \chi}} \right)^\tau$$

Currently there is little hope to relate the transcendental cohomological periods $\Omega_{(-1)^j \operatorname{sgn} \chi}$ to motivic periods in the sense of Deligne's Conjecture on special values in general for $n > 1$. Currently this is possible only in special cases where a link to Shimura varieties can be established, as is the case in [13]. We will discuss the hypothetical relation to Deligne's Conjecture in the next section.

5 Deligne's Conjecture

In this section we discuss the structure of the conjectural motives $M(\pi, \sigma)$ over F attached to (π, σ) , and also show that the periods from Theorem 4.13 are in accordance with the predictions from Deligne's Conjecture.

5.1 Motives

Attached to (π, σ) is a conjectural motive $M(\pi, \sigma)$ over F . It is characterized by the conjectural identity of L -functions (cf. [9])

$$L\left(s - \frac{2n-1}{2}, \pi \times \sigma\right) = L(s, M(\pi, \sigma)). \quad (49)$$

By recent work of Harris-Lan-Taylor-Thorne and independently Scholze, we know that we may attach a compatible system of ℓ -adic Galois representations

$$\rho_{\pi, \ell} : \operatorname{Gal}(\overline{F}/F) \rightarrow \operatorname{GL}_{n+1}(\overline{\mathbf{Q}}_\ell)$$

and

$$\rho_{\sigma, \ell} : \operatorname{Gal}(\overline{F}/F) \rightarrow \operatorname{GL}_n(\overline{\mathbf{Q}}_\ell)$$

to π and σ , outside a finite set of exceptional primes. These conjecturally provide us with the ℓ -adic realizations of the conjectural motives $M(\pi)$ and $M(\sigma)$, and $\rho_{\pi, \ell} \otimes \rho_{\sigma, \ell}$ is believed to be a compatible system of Galois representations as well, and conjecturally gives rise to the ℓ -adic realizations of $M(\pi, \sigma)$. We know that for unramified primes, the system is actually compatible. Therefore, replacing $M(\pi, \sigma)$ with this partially compatible system of Galois representations in (49), we obtain an identity of Euler products outside a finite set of exceptional primes (and possibly infinity), in a suitable right half plane.

Attached to $M(\pi, \sigma)$ is a collection of realizations. Apart from the ℓ -adic realizations that we already hinted at in the preceding discussion, most important for us are the Betti (or Hodge) and the de Rham realizations.

For each archimedean place v of F we have an associated embedding $v : F \rightarrow \mathbf{C}$, which gives rise to a base change $M(\pi, \sigma) \times_{F, v} \mathbf{C}$ of $M(\pi, \sigma)$, a motive over \mathbf{C} . This complex motive has a Betti realization,

$$H_{B, v}(M(\pi, \sigma)) := H_B(M(\pi, \sigma) \times_{F, v} \mathbf{C}),$$

which is a finite-dimensional \mathbf{Q} -vector space of dimension $d := (n+1)n$. Hodge theory provides us with a bigraduation $\{H_v^{p,q}\}_{p,q \in \mathbf{Z}}$ of the complexification

$$H_{B,v}(M(\pi, \sigma))_{\mathbf{C}} := H_{B,v}(M(\pi, \sigma)) \otimes_{\mathbf{Q}} \mathbf{C}.$$

The numbers

$$h_v^{p,q} := \dim H_v^{p,q}$$

are the Hodge numbers of $M(\pi, \sigma)$ relative to v . Now complex conjugation induces an involution

$$F_{\infty,v} : H_{B,v}(M(\pi, \sigma)) \rightarrow H_{B,v}(M(\pi, \sigma)).$$

It interchanges $H_v^{p,q}$ with $H_v^{q,p}$, and in particular $h_v^{p,q} = h_v^{q,p}$. It is believed that $M(\pi, \sigma)$ will be pure of a fixed weight $w \in \mathbf{Z}$, i.e.

$$h_v^{p,q} \neq 0 \implies p + q = w,$$

independently of v .

Furthermore we have the decomposition

$$H_{B,v}(M(\pi, \sigma)) = H_{B,v}^+(M(\pi, \sigma)) \oplus H_{B,v}^-(M(\pi, \sigma))$$

into (± 1) -eigenspaces of $F_{\infty,v}$. We write

$$d_v^{\pm} := \dim H_{B,v}^{\pm}(M(\pi, \sigma)).$$

We will see below that in our case those two dimensions agree.

The de Rham realization is an F -vectorspace $H_{\text{dR}}(M(\pi, \sigma))$ of dimension $(n+1)n$, and comes with a decreasing filtration F_{dR}^p , given by the hyper cohomology spectral sequence associated to the (motivic) de Rham complex. We write for the complexification relative to v ,

$$H_{\text{dR},v}(M(\pi, \sigma))_{\mathbf{C}} := H_{\text{dR}}(M(\pi, \sigma)) \otimes_{v,F} \mathbf{C},$$

and denote the induced filtration by $F_{\text{dR},v}^p$.

By GAGA we should have for each archimedean place v of F a comparison isomorphism

$$\iota_v : H_{B,v}(M(\pi, \sigma))_{\mathbf{C}} \xrightarrow{\sim} H_{\text{dR},v}(M(\pi, \sigma))_{\mathbf{C}},$$

which respects the Hodge decomposition and the Hodge filtration, i.e.

$$\iota_v \left(\bigoplus_{p' \geq p} H_v^{p',q'} \right) = F_{\text{dR},v}^p.$$

Following Deligne we set

$$H_{\text{dR}}^{\pm}(M(\pi, \sigma))_{v,\mathbf{C}} := H_{\text{dR}}(M(\pi, \sigma))_{\mathbf{C}} / F_v^{\mp},$$

where¹

$$F_v^\pm := F_{\text{dR},v}^{\frac{w+1}{2}}.$$

Counting dimensions, we see that we obtain the following refined comparison isomorphisms

$$\iota_v^\pm : H_{\text{B},v}^\pm(M(\pi, \sigma))_{\mathbf{C}} \xrightarrow{\sim} H_{\text{dR},v}^\pm(M(\pi, \sigma))_{\mathbf{C}}.$$

Since the left hand side has a natural \mathbf{Q} -structure, and the right hand side a natural F -structure, Deligne chooses basis in these rational structures and defines the periods

$$c_v^\pm(M(\pi, \sigma)) := \det(\iota_v^\pm)$$

with respect to these bases. Then $c_v^\pm(M(\pi, \sigma))$ is uniquely defined up to multiplication by elements in F^\times . We set for each finite order character ε of

$$\begin{aligned} \pi_0((F \otimes_{\mathbf{Q}} \mathbf{R})^\times) &= \pi_0(L), \\ c_\varepsilon &= \prod_v c_v^{\varepsilon(\mathbf{1}_v)}(M(\pi, \sigma)), \end{aligned} \tag{50}$$

and

$$d^\varepsilon = \sum_v d_v^{\varepsilon(\mathbf{1}_v)}.$$

5.2 Deligne's Conjecture for $L(s, M(\pi, \sigma) \otimes \chi)$

The L -functions associated to (π, σ) and $M(\sigma, \tau)$ conjecturally coincide up to translation and the latter has coefficients in the field of rationality $E := \mathbf{Q}(\pi, \sigma)$ of the pair π, σ , which is a number field [9], and contains the field $\mathbf{Q}(\mu, \nu)$.

In the sequel we consider $M(\pi, \sigma)$ as a motive over F with coefficients in E , i.e. in all realizations we extend scalars by tensoring with E . To keep the notation simple, we fix an embedding

$$E \rightarrow \mathbf{C}$$

once and for all. The statements that follow do not depend on the choice of embedding. The identity (49) actually is meaningless without a fixed choice of embedding. In this setting Deligne's Conjecture [11, Conjecture 2.8], reads in our case (cf. also (5.1.8) and section 8 in loc. cit., see also [4] for the behaviour under finite order character twists, as well as [3] for the periods attached to tensor products)

Conjecture 5.1 (Deligne). *For each $k \in \mathbf{Z}$ critical for $L(s, M(\pi, \sigma))$ and each finite order character $\chi : \text{Gal}(\overline{F}/F) \rightarrow \mathbf{C}^\times$ we have*

$$\frac{L(k, M(\pi, \sigma) \otimes \chi)}{G(\overline{\chi})^{d^{(-1)^k \varepsilon}} (2\pi i)^{kd^{(-1)^k \varepsilon}} c_{(-1)^k \text{sgn } \chi}} \in E(\chi).$$

¹In the next section we will see that w is odd in our case.

In the next section we will see in (51) that d^ε is independent of ε and thus the exponents of the Gauß sum $G(\overline{\chi})$ and of i in the numerator agree with those in Theorem 4.13. In light of (52) below the powers of π occuring in the numerator also agree.

5.3 Hodge numbers

The conjectural Hodge numbers attached to (π, σ) may be made explicit as follows. The position of $\pi_\infty \widehat{\otimes} \sigma_\infty$ in the Langlands classification of admissible representations of $G(\mathbf{R})$ is already determined by M , and thus by the pair (μ, ν) . This is made explicit in [9] for π and σ , and the data for the tensor product is calculated in Section 1.3 of [30].

With the notation of Theorem 4.13 we have $M = M(\mu)^\vee \otimes M(\nu)^\vee$ for absolutely irreducible rational G_{n+1} - and G_n -modules $M(\mu)$ and $M(\nu)$, of highest weights μ and ν , respectively. As before we consider the weights componentwise

$$\mu = \otimes_v \mu_v,$$

by archimedean places, where μ_v are dominant weights of GL_{n+1} , that we identify with integer tuples

$$\mu_v = (\mu_{v,1}, \mu_{v,2}, \dots, \mu_{v,n+1}),$$

satisfying the dominance condition

$$\mu_{v,1} \geq \mu_{v,2} \geq \dots \geq \mu_{v,n+1},$$

and the purity condition

$$\mu_{v,k} + \mu_{v,n+2-k} = w_1$$

with w_1 independent of the archimedean place v and $1 \leq k \leq n+1$. Likewise

$$\nu = \otimes_v \nu_v,$$

$$\nu_v = (\nu_{v,1}, \nu_{v,2}, \dots, \nu_{v,n}),$$

subject to the same dominance condition and the purity condition

$$\nu_{v,k} + \nu_{v,n+1-k} = w_2,$$

for all v and $1 \leq k \leq n$. Then the pair (w_1, w_2) has the same meaning as in (28). The critical s_0 for $L(s, \pi \times \sigma)$ are centered around

$$\frac{w_1 + w_2 + 1}{2},$$

which is a half integer, itself critical, due to our assumption on criticality, cf. [30, Proposition 2.2].

We see that for each archimedean place v of F the Hodge numbers $h_v^{p,q}$ vanish except in the following cases:

$$(p, q) = (\mu_{k,v} + \nu_{l,v} - k - l + (2n+1), \mu_{n+2-k,v} + \nu_{n+1-l,v} + k + l - 2), \quad 1 \leq k \leq n+1, 1 \leq l \leq n.$$

And in these cases we have

$$h_v^{p,q} = 1.$$

By Proposition 1.5 and Proposition 2.2 (and its corrected version as in the errata²) of [30] we see that $h_v^{p,q}$ gives the Hodge numbers of the conjectural motive $M(\pi, \sigma)$.

We see that

$$d_v^\pm = \frac{(n+1)n}{2},$$

and thus

$$d^\varepsilon = r_F \frac{(n+1)n}{2}, \quad (51)$$

independently of the signature ε . A more conceptual argument for this identity is the existence of an isomorphism

$$(\pi \hat{\otimes} \sigma) \otimes \text{sgn}_\infty \cong \pi \hat{\otimes} \sigma,$$

hence this symmetry must be reflected in the Langlands parameters, and thus also in the associated conjectural Hodge numbers.

By the above computation (see also Proposition 1.5 in [30] and also the formula given on p. 219 and p. 220 of loc. cit.), we also see that

$$\frac{L(s_1, \pi_\infty \times \sigma_\infty)}{L(s_0, \pi_\infty \times \sigma_\infty)} \in (2\pi)^{(s_0 - s_1)r_F \frac{(n+1)n}{2}} \mathbf{Q}^\times. \quad (52)$$

5.4 Compatibility with Deligne's Conjecture

Incorporating the results from the previous section into Theorem 4.13 we obtain

Theorem 5.2. *Assume $n \leq 2$ or that Conjecture 4.7 is true for $n > 2$. Let F be a totally real number field, and (π, σ) be a pair regular algebraic irreducible cuspidal automorphic representation of $\text{GL}_{n+1}(\mathbf{A}_F)$ and $\text{GL}_n(\mathbf{A}_F)$ respectively. Assume that $\pi_\infty \hat{\otimes} \sigma_\infty$ has non-trivial Lie algebra cohomology with coefficients in an irreducible rational $G_{n+1} \times G_n$ -module M , which we assume to be critical. Then there exist non-zero periods Ω_\pm , numbered by the 2^{r_F} characters \pm of $n\pi_0((F \otimes_{\mathbf{Q}} \mathbf{R})^\times)$, such that for each critical half integer $s_0 = \frac{1}{2} + j_0$ for $L(s, \pi \times \sigma)$, and each finite order Hecke character*

$$\chi : F^\times \backslash \mathbf{A}_F^\times \rightarrow \mathbf{C}^\times,$$

we have, in accordance with Deligne's Conjecture 5.1,

$$\frac{L(s_0, (\pi \times \sigma) \otimes \chi)}{G(\overline{\chi})^{\frac{(n+1)n}{2}} (2\pi i)^{j_0 r_F \frac{(n+1)n}{2}} \Omega_{(-1)^{j_0} \text{sgn } \chi}} \in \mathbf{Q}(\pi, \sigma, \chi).$$

Furthermore this expression is $\text{Aut}(\mathbf{C}/\mathbf{Q})$ -equivariant.

²which says that the map $(k, l) \mapsto (p, q)$ is injective whenever critical s exist.

Corollary 5.3. *Under the assumptions of Theorem 5.2 Deligne’s Conjecture 5.1 for the conjectural motive $M(\pi, \sigma) \otimes \chi$, for a rank 1 Artin motive χ , is equivalent to the statement*

$$\frac{\Omega_{(-1)^j \operatorname{sgn} \chi}}{(2\pi i)^{r_F \frac{(n+1)n^2}{2}} c_{(-1)^{j+n} \operatorname{sgn} \chi}} \in \mathbf{Q}(\pi, \sigma)^\times, \quad j \in \{0, 1\}.$$

The statement of Corollary 5.3 leaves out valuable finer structure, i.e. the conjectural implications that result from the fact that $M(\pi, \sigma)$ arises as a tensor product (cf. [3]), and also the finer description of Deligne’s periods c_\pm in terms of products of the periods $c_v^\pm(M(\pi, \sigma))$ attached to real embeddings as in (50) (cf. [4]). These finer results are known for Hilbert modular forms [18].

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